

## Usability Evaluation of Finite Element Method Equation Solvers

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### Introduction

The behaviour of a phenomenon in an engineering system depends upon the geometry or domain of the system, the property of the material or medium, and the boundary, initial and loading conditions. For an engineering system, the geometry or domain can be very complex. Further, the boundary and initial conditions can also be complicated. It is therefore, in general, very difficult to solve the governing differential equation via analytical means. In practice, most of the problems are solved using numerical methods. Among these, the methods of domain discretization championed by the finite element method are the most popular, due to its practicality and versatility.

The finite element method leads to a system of equations that must be solved simultaneously [1,2]. A complex model can generate a very large system of equations. Each equation represents an unknown quantity which is required to find. Each unknown quantity is also referred to as a degree of freedom. Solving a large system of simultaneous equations requires a long time and large computer resources.

Software package ANSYS (ANSYS Inc., www.ansys.com) has industry leading solver technology to support robust and comprehensive simulation capability. Two types of direct solvers (Sparse Direct Solver, Frontal (Wavefront) Solver) are implemented in it and also three iterative solvers which are based on the conjugate gradient method (Jacobi Conjugate Gradient solver, Preconditioned Conjugate Gradient solver, Cholesky Conjugate Gradient solver) [3–6].

### Modeling algorithm

In order to determine the performance of mentioned solvers the spatial model of colour TV-tube symmetrical electronic optical system is used [7,8]. Model is created only for side presumable electron trajectory. Such decision to use this system was assumed because electronic optical system modeling technique is already highly elaborated.

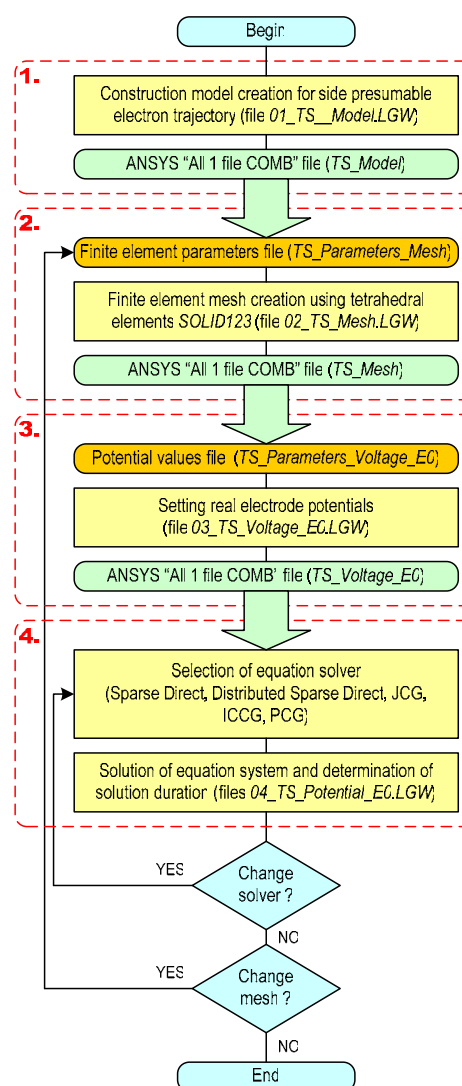


Fig. 1. General modeling algorithm

EOS modeling consists of four main stages [7,8] construction model creation, finite element mesh generation (tetrahedral finite element *SOLID123* is used), setting of electrode potentials and solution of equation system (Fig. 1).

The performance of Sparse Direct, Distributed Sparse Direct, Jacobi Conjugate Gradient, Preconditioned Conjugate Gradient, Cholesky Conjugate Gradient solvers is analyzed during modeling.

### System of linear equations

The system of linear equations generated by the finite element technique is solved using a direct elimination or an iterative method [3]. A direct elimination method is based on a Gaussian elimination method, according to which the values of the unknown vector of variables  $\{u\}$  are found in the following equation

$$[K]\{u\} = \{F\}; \quad (1)$$

where  $[K]$  – conductivity matrix;  $\{u\}$  – vector of nodal unknown values;  $\{F\}$  – applied load vector.

The direct elimination method [3] consists of decomposition of the matrix  $[K]$  into lower and upper triangular matrixes  $[K] = [L][U]$ . Then forward and backward substitutions using matrixes  $[L]$  and  $[U]$  are made in order to compute the vector  $\{u\}$ , containing the values of solution results.

In case of iterative method [3] the initial guess is made for the value  $\{u\}_1$  of the solution vector  $\{u\}$  and then a iterative steps are performed resulting in a sequence of vectors  $\{u\}_2, \{u\}_3, \dots, \{u\}_n$ , such that, in the limit,  $\{u\}_n = \{u\}$  as  $n$  tends to infinity. During the calculation of vector  $\{u\}_{n+1}$  matrix  $[K]$ , vectors  $\{F\}$  and  $\{u\}$  are used from one of the previous iterations. Typically the solution converges to within a specified tolerance after a finite number of iterations. Further we will present a short overview of these methods [3].

### Sparse Direct Solver

Comprehensive description of Sparse Direct Solver used in ANSYS/Emag is given in [3]. The linear matrix equation (1) is solved by triangular decomposition of matrix  $[K]$  to yield the following equation:

$$[L][U]\{u\} = \{F\}; \quad (2)$$

where  $[L]$  – lower triangular matrix;  $[U]$  – upper triangular matrix.

By substituting

$$\{w\} = [U]\{u\} \quad (3)$$

it is possible to obtain  $\{u\}$  by first solving the triangular matrix system for  $\{w\}$  by using the operation of direct insertion of (3) into the (2) equation

$$[L]\{w\} = \{F\} \quad (4)$$

and then vector of unknown quantities  $\{u\}$  is calculated using the back substitution operation on a triangular matrix given by:

$$[U]\{u\} = \{w\}. \quad (5)$$

When the matrix  $[K]$  is symmetric, the description given above could use the substitution:

$$[K] = [L][L]^T. \quad (6)$$

However, it is modified as:

$$[K] = [L][D][L]^T; \quad (7)$$

where  $[D]$  – a diagonal matrix.

The diagonal terms of matrix  $[D]$  may be negative in the case of some nonlinear finite element analysis. This allows the generation of matrix  $[L']$  without the consideration of a square root of negative number. Thus, equations (2–5) become:

$$[L'][D][L']^T \{u\} = \{F\}, \quad (8)$$

$$\{w\} = [D][L']^T \{u\}, \quad (9)$$

$$[L']\{w\} = \{F\}, \quad (10)$$

$$[D][L']^T \{u\} = \{F\}. \quad (11)$$

Since the matrix  $[K]$  is normally sparsely populated with coefficients dominantly located around the main diagonal, the Sparse Direct Solver is intended to handle only the non-zero entries in the matrix  $[K]$ . In general, during the Cholesky decomposition of the matrix  $[K]$  shown in equation (2) or (8), non-zero coefficients appear in matrixes  $[L]$  or  $[L']$  at coefficient locations where  $[K]$  matrix had zero entries. By using the Sparse Direct Solver method this fill-in of the matrix is minimized by reordering the equation numbers in the matrix  $[K]$ .

### Frontal Solver

Comprehensive description of Frontal Solver used in ANSYS/Emag is given in [3]. The computer time required in order to solve the problem is proportional to the square of the mean wavefront magnitude. The wavefront magnitude is determined by the order in which the elements are arranged. The node numbers of all elements are verified in order to determine which element is the last to use for each node. Since the overall system of equations is formed from the element matrixes, the equations for a node which occurs for the last time are algebraically solved in terms of the remaining unknown values and eliminated from the formed matrix by using the Gauss elimination technique. The active equations are represented by:

$$\sum_{j=1}^L K_{kj} u_j = F_k; \quad (12)$$

where  $K_{kj}$  – stiffness coefficient relating the force at the degree of freedom  $k$  to the displacement at the degree of freedom  $j$ ;  $u_j$  – nodal displacement of the degree of freedom  $j$ ;  $F_k$  – nodal force of the degree of freedom  $k$ ;  $k$  –

row number;  $j$  – column number;  $L$  – number of equations (rows).

To eliminate an equation, for which  $i = k$ , the equation is normalized in the following way

$$\sum_{j=1}^L \frac{K_{ij}}{K_{ii}} u_j = \frac{F_i}{K_{ii}}. \quad (13)$$

Equation (13) can be written as:

$$\sum_{j=1}^L K_{ij}^* u_j = F_i^*, \quad (14)$$

where  $K_{ij}^* = \frac{K_{ij}}{K_{ii}}$ ,  $F_i^* = \frac{F_i}{K_{ii}}$ ,  $K_{ii}$  – is known as the “pivot” [3]. If the absolute value of  $K_{ii}$  is sufficiently small, it is mathematically equal to zero. This usually means the structure is insufficiently constrained.

Equation (14) is written to a file for later back-substitution. The remaining equations are modified as:

$$K_{kj}^* = K_{kj} - K_{ki} K_{ij}^*, \quad (15)$$

$$F_k = F_k - K_{ki} F_i^*, \quad (16)$$

where  $k \neq i$  so that

$$\sum_{j=1}^{L-1} K_{kj}^* u_j = F_k^*, \quad (17)$$

where  $k$  varies from 1 to  $L-1$ . When the row  $i$  is eliminated from the equation (17), this procedure is repeated for all other equations suitable for elimination.

The equations for a node which occurs for the first time are added to the formed matrix over the course of the solution process. So, the formed matrix expands and contracts as node make their first and last appearance in the element definitions. The varying size of the active matrix is considered as the momentary wavefront size.

When several elements are connected to the same node, the degrees of freedom related to these elements remain active until the wavefront “passes” all elements connected to the node. Degrees of freedom related by constraint equations or coupled nodes remain active until the wavefront “passes” all elements connected to the related degrees of freedom. Master degrees of freedom remain active in memory and are not deleted from the wavefront. This procedure is extensively described in [3]. The ideal model size when using Frontal Solver is under fifty thousand degrees of freedom, therefore it is not suitable for three-dimensional modeling of electronic optical system.

### Iterative Solvers

The iterative solvers are based on the conjugate gradient method. In this case the system of linear equations (1) is also solved. In the conjugate gradient method, the solution is found as a series of vectors  $\{p_i\}$ :

$$\{u\} = \alpha_1 \{p_1\} + \alpha_2 \{p_2\} + \dots + \alpha_m \{p_m\}, \quad (18)$$

where  $m$  is no larger than the matrix size  $n$ .

The rate of convergence of the conjugate gradient algorithm is proportional to the square root of the conditioning number of the matrix  $[K]$  where the condition number of matrix  $[K]$  is equal to the ratio of the maximum eigenvalue of  $[K]$  to the minimum eigenvalue of  $[K]$ . A preconditioning procedure is used to reduce the condition number of linear equations (1). In the Jacobi Conjugate Gradient algorithm, the diagonal elements of the matrix  $[K]$  are used as the preconditioner matrix  $[Q]$ , while in the Cholesky Conjugate Gradient and Preconditioned Conjugate Gradient algorithms, a more sophisticated preconditioner matrix  $[Q]$  is used. The conjugate gradient algorithm with preconditioning is widely described in [3].

Convergence is achieved when:

$$\frac{\{R_i\}^T \{R_i\}}{\{F\}^T \{F\}} \leq \varepsilon^2; \quad (19)$$

where  $\varepsilon$  – user supplied tolerance;  $\{R_i\} = \{F\} - [K]\{u_i\}$ ,  $\{u_i\}$  – solution vector at iteration  $i$ . It is assumed that the initial starting vector  $\{u_0\}$  is a zero vector.

### Analysis of solver performance

The designed algorithm used to determine solver performance is presented in Fig. 1. This algorithm differs from earlier algorithms [7,8] in the possibility to select the particular solver and also calculation of electric potential at specifically selected points of trajectory cylinders is not calculated (only values of electric potential at the nodes of finite element mesh are calculated).

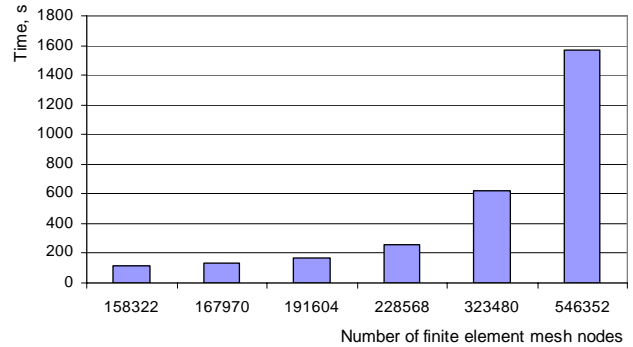


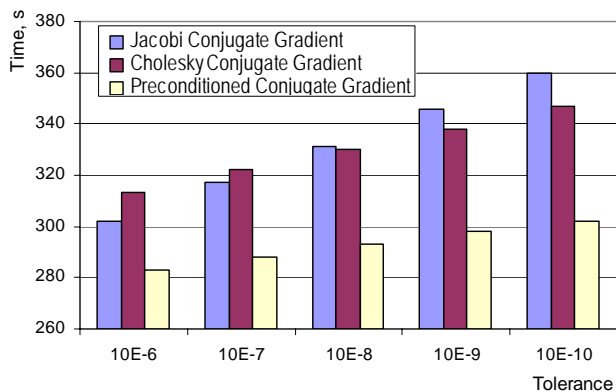
Fig. 2. Sparse Direct Solver performance dependence on the number of finite element mesh nodes

In order to determine the performance (solution duration) of solvers the electronic optical system model containing 394288 tetrahedral finite elements *SOLID123* (546352 nodes) was used. Mesh is generated for one time and is used for analysis of performance of all solvers.

In case of the Sparse Direct Solver the modeling performance dependence on the number of finite elements (finite element nodes) was also determined (Fig. 2), and in case of iterative solvers – the performance dependence on the selected tolerance (Fig. 3).

The personal computer with the following characteristics was used for computations: mainboard –

Biostar M7VIT Pro; CPU – AMD Athlon XP 2500+ (1.83 GHz); RAM – 1.5 GB DDR-SDRAM (memory type PC3200U and timings 2.5-4-4-8); HDD – WDC WD600JB; OS – Windows XP Professional (Win32 x86) 5.01.2600 (Service Pack 2). Received results are presented in Fig. 2 and Fig. 3.



**Fig. 3.** Iterative solver performance dependence on the tolerance level

After analyzing solver performance it was found, that it is most purposeful to use Preconditioned Conjugate Gradient solver to perform the spatial modeling of electronic optical system, because it is 5 times faster than Sparse Direct Solver, and approximately 7% faster than Jacobi Conjugate Gradient and Cholesky Conjugate Gradient solvers.

## Conclusions

Longest time to solve the system of equations is taken when using Sparse Direct and Distributed Sparse Direct solvers (approximately five times longer than using iterative solvers).

The performance of iterative solvers Jacobi Conjugate Gradient, Cholesky Conjugate Gradient and Preconditioned Conjugate Gradient differs marginally,

approximately by 10%. In this case the selection of solver is mainly determined the amount of available memory.

The defined tolerance has little impact on the performance of Jacobi Conjugate Gradient, Cholesky Conjugate Gradient and Preconditioned Conjugate Gradient solvers. The difference of performances when tolerances 10E-6 and 10E-10 are set when using Jacobi Conjugate Gradient solver are 15%, Cholesky Conjugate Gradient solver – 10%, Preconditioned Conjugate Gradient solver – 7%.

Sparse Direct Solver is most effective to use when the number of degrees of freedom is approximately half million, and for Jacobi Conjugate Gradient, Preconditioned Conjugate Gradient, Cholesky Conjugate Gradient solvers this number is approximately ten million degrees of freedom.

## References

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**P. Tarvydas, A. Noreika.** Usability Evaluation of Finite Element Method Equation Solvers // *Electronics and Electrical Engineering.* – Kaunas: Technologija, 2007. – No. 2(74). – P. 13–16.

Direct and iterative solution techniques of equation systems used in finite element method are analyzed. Comparison of these techniques, their advantages and drawbacks are characterized and recommendations for their applications are presented. Methods are analyzed by performing the modeling of the symmetrical electronic optical system using software package ANSYS/Emag. Electronic optical system modeling algorithm and technique was designed. After the modeling the performances of equation system solution methods were compared against each other. Ill. 3, bibl. 8 (in English; summaries in English, Russian and Lithuanian).

**П. Тарвидас, А. Норейка.** Оценка возможностей применения методов решения систем уравнений, используемых в методе конечных элементов // *Электроника и электротехника.* – Каунас: Технология, 2007. – № 2(74). – С. 13–16.

Анализируются прямые и итерационные методы решения систем уравнений, используемые в методе конечных элементов. Представлено сравнение различных методов, обсуждаются их преимущества и недостатки, представлены рекомендации для их использования. Методы анализируются, применяя моделирование симметрической электронной оптической системы, используя программный пакет анализа методом конечных элементов ANSYS/Emag. Создан алгоритм и методика моделирования электронной оптической системы. После моделирования было сравнено между собой быстродействие различных методов. Илл. 3, библи. 8 (на английском языке; рефераты на английском, русском и литовском яз.).

**P. Tarvydas, A. Noreika.** Baigtinių elementų metode taikomų lygčių sistemų sprendimo metodų pritaikymo galimybių įvertinimas // *Elektronika ir elektrotechnika* – Kaunas: Technologija, 2007. – Nr. 2(74). – P. 13–16.

Nagrinėjami baigtinių elementų metode naudojami tiesioginiai ir iteraciniai lygčių sistemų sprendimo metodai. Pateikiamas įvairių metodų palyginimas, apibūdinamos jų teigiamybės ir trūkumai, pateikiamos jų taikymo rekomendacijos. Metodai analizuojami atliekant simetrinės elektroninės optinės sistemos modeliavimą baigtinių elementų analizės programa ANSYS/Emag. Sudarytas elektroninės optinės sistemos modeliavimo algoritmas ir metodika. Atlikus analizę palyginta įvairių metodų greitaveika. Il. 3, bibl. 8 (anglų kalba; santraukos anglų, rusų ir lietuvių k.).