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On Numerical Approach to Non-Markovian Stochastic Systems Modeling

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Abstract: The paper considers the problem of representing non-Markovian systems that evolve stochastically over time. It is often necessary to use approximations in the case the system is non-Markovian. Phase type distribution is by now indispensable tool in creation of stochastic system models. The paper suggests a method and software for evaluating stochastic systems approximations by Markov chains with continuous time and countable state space. The performance of a system is described in the event language used for generating the set of states and transition matrix between them. The example of a numerical model is presented.

Key words: Non-Markovian system approximation, phase type distribution, Markov chain, numerical model.

1. Introduction

A problem in system modeling having a wide range of important practical applications arises when the system is inherently stochastic and complete statistical description is not known. In order to evaluate system performance, a mathematical model must be developed. As the system is random in nature, a statistically based model is required.

Stochastic system models are often based on continuous-time Markov processes. Markov chains are commonly used for stochastic systems modeling.

However, some important aspects of system behavior cannot be easily captured in a Markov model. Very often the life-times distributions connected with a system are simply not exponential. For example, electronic component failure times often approximately follow a Weibull or lognormal distribution [1]. When an exponential distribution is both unrealistic and unsatisfactory for representing life-time distribution, then the usual approach is to use the "method of (exponential) stages" [2-6]. This method is both general and compatible with definition of Markov processes. It is general in what the authors can represent arbitrary distributions arbitrary closely. It is compatible with Markov processes because the only memory introduced is the distribution stage to accommodate this additional memory the authors refine their state definition.

It is known that creation of analytical models requires large efforts. Use of numerical methods permits to create models for a wider class of systems. The process of creating numerical models for systems described by Markov chains consists of the following stages: (1) definition of the state of a system; (2) definition of the set of events that can occur in the system; (3) generating the states of the system and infinitesimal generator matrix; (4) creating equations describing Markov chain; (5) computation of stationary probabilities of Markov chain; (6) computation of the performance measures of the system. The most

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difficult stages are obtaining the set of all the possible states of a system and transition matrix between them. A method for automatic construction of numerical models for systems described by Markov chains with a countable space of states and continuous time is proposed in the paper.

To construct a model, it is needed to describe the performance of a stochastic system in the event language [7-9]. It allows automating some stages of the model. The created software in C++ generates the set of possible states, the matrix of transitions among states, constructs and solves equilibrium equations to find steady state probabilities. The paper is organized as follows: Section 2 introduces the description of the behavior of a stochastic system; section 3 is given the example; section 4 presents conclusions.

2. Description of the Behavior of a Stochastic System by Markov Processes

Consider a stochastic system with a random state vector at time *t*:

$$Y(t)=\left(Y_1(t),Y_2(t),\dots,Y_m(t)\right)$$

where $Y_i(t),\ i=\overline{1,m}$ are discrete random variables. Assume that the system can change a state at any time. If all the life-times connected with a system have exponential distribution, the process is Markovian and standard methods yield a set of ordinary linear differential equations to determine the behavior of the process and a set of simple linear equations to determine the equilibrium distribution, if one exists. Examples of life-times are the service-times in a queue, the intervals between the arrivals of successive customers in a queue, the division times of bacteria, and so on. Markov modelling is a common approach to analyzing the performance of various stochastic systems.

Thus, many real world systems can be modeled by Markov chain $\{Y(t),\ t\geq 0\}$ with countable space of states and continuous time. The authors denote $B=\{S_j,\ j\geq 1\}$ where $S_j=(y_{1j},y_{2j},\dots,y_{mj})$ is the state space and $P\{Y=S_j\}=\pi_j,\ j\geq 1$ is the probability of *j*-th state. The steady state probabilities can be calculated by solving the following system of equations:

$$\nu_j\pi_j=\sum_{k\neq j}\pi_k\lambda_{kj},\ S_j\in B$$

where $\sum_{S_j\in B}\pi_j=1$, λ_{kj} is a transition rate from state *k* to state *j* and $\nu_i=\sum_{j\neq i}\lambda_{ij},\ S_i\in B$. Based on the calculated probabilities, a wide range of relevant performance characteristics of system under investigation can be computed.

Formally, a system can be depicted by the relation $S=\{X,R\}$, where $X=\{X_1,X_2,...,X_n\}$ is the set of input elements into the system; *R* describes the behavior of the system, restrictions and control. The mechanism of input-output is needed to know as to describe the performance of the system. In other words, for given *X* and *R*, the vector $Y=\{Y_1,Y_2,...,Y_m\}$, is needed to find out, i.e., to know the depiction $f:\{X,R\}\to Y$, i.e., for given $\{X,R\}$, it is possible to calculate Y, if the depiction f is known. In such a way, deterministic can be described as stochastic systems. For deterministic systems given *X*, *R* and *f* the output vector *Y* is obtained with probability one. For stochastic systems, the output vector *Y* is a random value with distribution

$$P(Y=(y_{1j},y_{2j},...,y_{mj})\,|\,X,R,f)=\pi_j,\ j=1,2,...$$

if the random value *Y* is discrete.

Assume that a system works in stationary mode and states of the system change by influence of *n* events stream. The interval X_i between two successive events is a random variable with distribution function $G_i(x)$, *i* = 1, *n*. So the set of input elements $X=(X_1,X_2,...,X_n)$ consists of random variables. The output is a system state $Y=(Y_1,Y_2,\dots,Y_m)$. Now attention is turned to systems with non-Markovian event processes. In the absence of the memoryless property, the residual lifetimes of all events must also be remembered. Let the observed current state be *Y*, with a feasible event set *E*(*Y*). Each event $e_j\in E(Y)$ has a clock value (residual lifetime) Z_j. Then, the probability that the triggering event is some e_j is given by the probability that event has the smallest clock value among all events in $E(Y)$:

$$P(e_j) = P\left[Z_j = \min_{e_i \in E(Y)} \{Z_i\} \right]$$

To determine this probability, generally information on the random variables Z_j is needed. It is only in the case of Markov chains the memoryless property allowed to obtain:

$$P(e_j) = \frac{\lambda_j}{\Lambda(Y)}$$

where λ_j is the exponential rate of event e_j and

$$\Lambda(Y) = \sum_{e_i \in E(Y)} \lambda_i$$

In this case, no information on the random variables Z_j is needed.

Let $e_1, e_2, ..., e_k$ be collection of exponential events with a common fixed rate, which the authors intend to use as building blocks. Then, a new event e may be defined as occurring only after all events $e_1, e_2, ..., e_k$ have occurred in series, one immediately following the other. By adjusting the number k of building blocks used, a variety of events e with different lifetime characteristics can be generated. Alternatively, e may be defined as occurring only after some event $e_i, i = 1, ..., k$ has occurred, where e_i is chosen randomly with probability p_i. Again, by adjusting the probabilities p_i and the number k, different event processes could be generated. In this way, an event e is decomposed into stages, each stage represented by building-block event e_i. The idea here is to preserve the basic Markovian structure (since all building-block events are Markovian). At the expense of larger state space (since the authors will be forced to keep track of the stage the system is in, in order to be able to tell when e does in fact occur). The suggested method of constructing distribution is called the phase-type distribution or a mixture of exponential distributions.

A popular approach in mapping a general probability distribution, G, into a phase type (PH) distribution, P, is to choose P such that some moments of P and G agree. To obtain accurate results, it is desirable to match as more as possible moments of the input distribution G by P. Matching more moments may be possible if many exponential distributions (phases) are allowed to use. However, the use of many phases in the PH increases the complexity of the Markov chain, and makes its analysis hard. Matching many moments using a small number of phases may be possible if unbounded computational resources are allowed to use or if the class of input distributions is limited. However, these limitations are not desirable. Achieving all the four desirable properties is the challenge in designing a moment matching algorithm [3]:

(1) It is desirable that more moments of the input distribution, G, and matching PH distribution, P, agree;

(2) It is desirable that P have a small number of phases;

(3) It is desirable that the algorithm be defined for a broad class of input distributions;

(4) It is desirable that algorithm have short running time.

The general approach in designing moment matching algorithms in the literature is to start by defining a subset S of PH distributions, and then map each input distribution into a distribution in S.

The following is m non-intersected sets (building blocks):

$$A_k = \left\{ p_i^{(k)},\ \mu_i^{(k)} e^{-\mu_i^{(k)} x},\ i = \overline{1, n_k} \right\}$$

$$\sum_{i=1}^{n_k} p_i^{(k)} = 1,\ k = \overline{1, l}$$

where $\mu_i^{(k)} e^{-\mu_i^{(k)} x}$—exponential probability distribution. From the sets A_k, $k = \overline{1, l}$, the desired structure of exponential stages can be constructed. The Laplace transform gives some function $F(s)$. It is shown that the poles of the function may be complex numbers. Since $F(s)$ is always real then the complex poles must be present in conjugated pairs and real parts of the poles must be negative. Then despite of the fact that $\{\mu_i\}$ are complex, it may be formally possible to investigate the process as Markovian [10].

To find the unknown parameters of exponential stages, the system of equations must be solved. Equating the first initial moments νi of the original $G(x)$ and the approximating density

$$\nu_i = (-1)^i F^{(i)}(s)\big|_{s=0},\ i = 1, 2, \ldots, j$$

and adding the condition of the zero initial moments equality to 1, the authors receive a system of nonlinear algebraic equations for the unknown parameters determination; j is the number of unknown parameters.

Life-time T has a general probability distribution function $G(t)$. Useful approximation can be obtained by the mixture of exponential (phase-type) distributions [9]. Suppose that $ET = m$ and $DT = \sigma^2$. The function $G(t)$ approximate by the distribution of a random variable represented as

$$X = \begin{cases} X_1 + X_2 & \text{with probability} \quad p \\ X_1 & \text{with probability} \quad 1-p \end{cases}$$

where X_1 and X_2 are independent random variables having exponential distributions with respective means $1/\mu_1$ and $1/\mu_2$. In words, the life-time X first goes through an exponential phase X_1 and next it goes through a second exponential phase X_2 with probability p or it goes out with probability 1-p.

Approximation of general distribution G using phase-type distribution is depicted in Fig. 1.

The distribution function of variable X is given by

$$F(x) = 1 - e^{-\mu_1 x} + \frac{p\mu_1}{\mu_2 - \mu_1}(e^{-\mu_2 x} - e^{-\mu_1 x})$$

where

$$\mu_{1,2} = \frac{1}{(\nu^2+1)m} \pm \frac{\sqrt{2\nu^2+1}}{(\nu^2+1)m} \cdot i,\ p = 1,\ i^2 = -1,\ \nu^2 = \frac{\sigma^2}{m^2}$$

if $\nu^2 < \frac{1}{2}$, and $\mu_1 = \frac{2}{m},\ \mu_2 = \frac{1}{m\nu^2},\ p = \frac{1}{2\nu^2}$, if $\nu^2 \geq \frac{1}{2}$.

After approximation, a new system, isomorphic to the original system is obtained. So the authors could expand the class of stochastic systems which are possible to model by Markov chains.

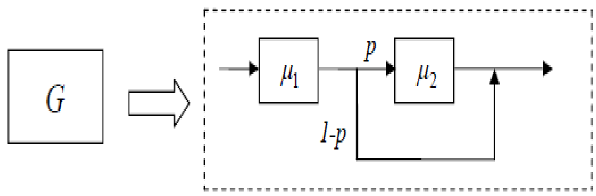

Fig. 1 Approximation of a general function.

It is known that creation of analytical models requires large efforts. Use of numerical methods permits to create models for a wider class of systems. The process of creating numerical models for systems described by Markov chains consists of the following stages: (1) definition of the state vector $Y = (Y_1, Y_2, \ldots, Y_m)$; (2) the set of events $E = (e_1, e_2, \ldots, e_k)$ which may occur in the system; (3) creating equations describing Markov chain; (4) computation of stationary probabilities of Markov chain; (5) computation characteristics of the system performance. The most difficult stages are obtaining the set of all the possible states of a system and transition matrix between them.

3. Example

Apply the described technique to construct a model of a queuing system with priorities.

Consider the system $S = \{X, R\}$ with the depiction:

$$f: \{X, R\} \to Y$$
$$X = (X_{1j}; X_{2j},\ j = \overline{1,k})$$
$$R = (NPRP; Q \le l_j,\ j = \overline{1,k}; FCFS)$$
$$Y = (Y_1, \ldots, Y_{k+2})$$

where the random variables X_{1j} describe the input flows of customers with preemptive priorities and X_{2j} represent respective service times.

The stochastic system is the queuing system with only one service node. The waiting rooms in queues Q_j are restricted by numbers l_j. The inputs of customers are Poisson $M(\lambda_j)$ with rates $\lambda_j, j = \overline{1,k}$ and respective service times are distributed by the following deterministic distribution functions:

$$D_j(x) = \begin{cases} 0, & x \le d_j \\ 1, & x > d_j \end{cases}$$

The service strategy is FCFS (first come first service).

Approximate the distribution functions $D_j(x)$ by the functions:

$$G_j(x) = 1 - e^{\mu_1^{(j)} x} + \frac{p\mu_1^{(j)}}{\mu_2^{(j)} - \mu_1^{(j)}}\left(e^{\mu_2^{(j)} x} - e^{\mu_1^{(j)} x}\right)$$

where

$$p=1;\ \mu_1^{(j)}=\frac{1}{d_i}+\frac{1}{d_i}\cdot i;\ \mu_2^{(j)}=\frac{1}{d_i}-\frac{1}{d_i}\cdot i,\ i^2=-1$$

The performance of the queuing system can be described by Markov process with countable state space and continuous time.

The set of possible events in the system is as the following:

$$E=\left\{e_{1j},\ j=\overline{1,k}, e_2, e_3\right\}$$

where:

e_{1j} —arrived the a customer from j-th flow, $j=\overline{1,k}$;

e_2 —a customer is served in the first phase;

e_3 —a customer is served in the second phase.

The state of system is

$$Y=\left\{Y_j,\ j=\overline{1,k}, Y_{k+1}, Y_{k+2}\right\}$$

where:

Y_j —the number of customers from j-th flow;

$$Y_{k+1}=\begin{cases}0, & \text{if the system is empty}\\ \text{j}, & \text{if the customer is served from the jth flow}\end{cases}$$

$$Y_{k+2}=\begin{cases}0, & \text{if the system is emty}\\ 1, & \text{if the customer is served in the first phase}\\ 2, & \text{if the customer is served in the second phase}\end{cases}$$

The number N of all the possible states in the system is equal to

$$N=2k\prod_{i=1}^{k}\left(l_i+1\right)+1$$

Denoted by $\pi_i,\ i=\overline{1,N}$ the steady state probabilities of the states; $L_q^{(j)}$ is the average number of customers in j-th queue; $W_q^{(j)}$ is the average waiting time in j-th queue. The denoted performance characteristics of the system are calculated by the formulas:

$$L_q^{(j)}=\sum_{y_j=1}^{l_j}\sum_{y_1,\dots,y_{k+2}} y_j\pi(y_1,\dots,y_j,\dots,y_k,y_{k+1},y_{k+2})$$

$$W_q^{(j)}=L_q^{(j)}/\lambda_j,\ j=\overline{1,k}$$

If the waiting rooms are not bounded, then the same characteristics may be computed analytically by the formulas [8]:

$$W_q^{(j)}=\frac{\sum_{m=1}^{k}\lambda_m\left(E_m^2(t)+\sigma_m^2(t)\right)}{2(1-S_{j-1})(1-S_j)}$$

$$L_q^{(j)}=\lambda_j W_q^{(j)}$$

$$S_0\equiv 0,\ S_j=\sum_{i=1}^{j}\rho_i<1,\ \rho_i=\lambda_i E_i(t),\ j=\overline{1,k}$$

$E_j(t)$ and $\sigma_j^2(t)$ are the service time means and dispersions.

The system was modeled with the following parameters:

$$k=2,\ \lambda_1=4,\ \lambda_2=3,$$
$$d_1=10^{-1},\ d_2=9^{-1},\ l_1=7,\ l_2=11$$

For computing the performance characteristics the complex probabilities were used.

The results of numerical model are the following:

$$L_q^{(1)}=0.25674,\quad L_q^{(2)}=0.72127,$$
$$W_q^{(1)}=0.064185,\quad W_q^{(2)}=0.2404233$$

The results of analytical model are

$$L_q^{(1)}=0.25679,\quad L_q^{(2)}=0.72132$$
$$W_q^{(1)}=0.0641975,\quad W_q^{(2)}=0.24044$$

4. Conclusions

The results showed that a non-Markovian queuing system with infinite number of states can be approximated by Markovian model with finite number of states with required accuracy. The complex probabilities may be used. The proposed approach for modeling non-Markovian systems can be applied for modeling financial markets and queuing systems.

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