625. Formation of structural matrices for finite elements of piezoceramic structures

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Abstract. This paper deals with the description of a theoretical background of systematic computer algebra methods for the formation of structural matrices of piezoceramic finite elements. Piezoceramic actuators are widely used for high-precision mechanical systems such as positioning devices, manipulating systems, control equipment, etc. In this paper, the efficiency of computer algebra application was compared with the numerical integration methods of formation of the structural matrices of the finite elements. Two popular finite elements are discussed for modeling piezoceramic actuators: sector type and the triangular one. All structural matrices of the elements were derived using the computer algebra technique with the following automatic program code generation. Due to smaller floating point operations, the computer time economy is followed by an increased accuracy of computations, which is the most important gain achieved in many cases.

Keywords: computer algebra, structural matrices, piezoactive materials, piezoceramic structures.

1. Introduction

Recent advances in the development, theory and applications of new smart materials, structures and devices, including the materials with extremely high piezoelectric or magnetostrictive properties have extended the area of mechatronics, providing systems with very high levels of integration and multifunctionality [3]. Some concepts, e.g., actuators with an infinite number of degrees of freedom, transmission of energy to actuators through some distance, active bearings, etc., could be described as solutions looking for a problem (Fig. 1). In some cases the introduction of piezomechanical systems creates a new synergistic effect and, as in all the integrated systems, the problem of maximum interaction between subsystems is the key to optimum design. Although the introduction of new piezoactive materials, which have found application in the areas of actuators, transducer technology, energy transformers, control devices, etc., has been very intensive in recent years [1,7,13,17], the main concepts, ideas and effects are relatively unknown to design engineers. Describing the holistic complex interaction of dynamic effects, energy transformation and devices based on them, while the physics involved in piezoelectric theory may be regarded as a coupling between Maxwell's equations of electromagnetism and the elastic stress equations of motion [10,12,14,15].

The finite element method (FEM) is widely used for modeling of complicated structures [18]. In the FEM the continuum is digitized and thus the numerical approximation is chosen.

A result of sufficient accuracy can often be obtained by numerical integration, and with more effective computer facilities this technique is most common. However, the order of numerical integration implies a thorough adjustment to the order of polynomial expressions to be integrated, and this fact is not generally known. The use of a more precise formula for numerical integration will not produce more accurate results, but will sometimes 'stiffen' the examined structure [5,6]. On the other hand, even the use of an exact numerical integration scheme always yields some numerical errors, which may become important in particular finite element applications, requiring a high accuracy of analysis [11,16].

Moreover, it is known that the formation of structural matrices is one of the most computationally expensive procedures in FEM [1], and the use of numerical integration for these purposes is not the optimal solution. This fact becomes particularly keen in the finite element applications in structures of piezoactive materials. In addition to the conventional stiffness matrix, there appear three additional matrices of mass, electro elasticity and capacity [1, 4]. The numerical integration procedure for the formation of the mentioned matrices is especially expensive, because the power of integral expressions is higher. Computer algebra offers wide opportunities in forming structural matrices of finite element. It is a pity to observe that a lot of routine algebraic operations in FEM is usually conducted by hand or with the help of numerical approximations, but that could be accomplished by means of the computer algebra technique in a faster, more reliable and smarter manner. Another key aspect is that the results, obtained in this way, often give a remarkable economy of computer resources and assure the proper accuracy of evaluations [5]. Despite that, even the classical books on FEM [18] do not mention the possibility of symbolic manipulations in the formation of structural matrices.

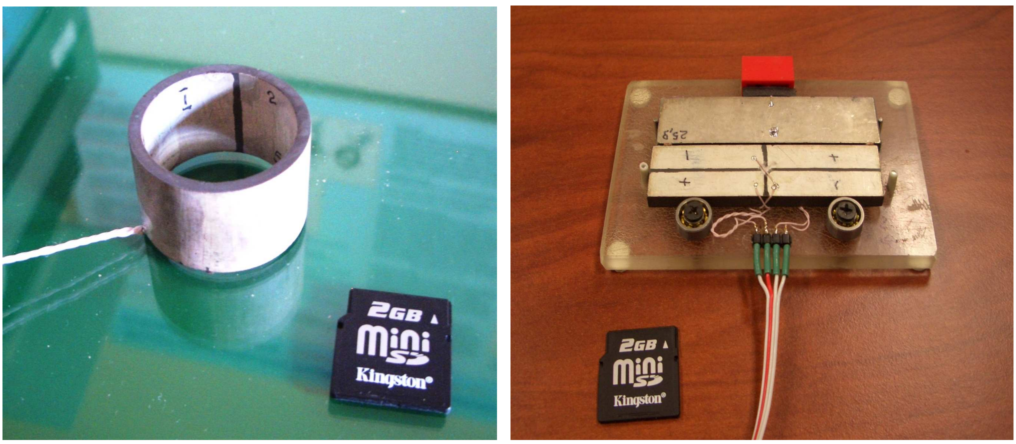

Fig. 1. Developed piezoceramic actuators

There is one advantage of numerical integration mentioned: for a particular type of finite elements, the structural matrices should always be expressed in the same way through interpolating functions and their derivatives. Therefore the universal computational subroutines can be applied to various finite elements [9, 18]. However, the same extent of generality is a characteristic feature of computer algebra methods, too. Still more, the user of programs, generated by computer algebra does not need to consider the powers of polynomial expressions to be integrated, which is necessary for selecting a proper numerical integration scheme [2]. Even the users of modern commercial codes (such as COMSOL, ANSYS, etc.) have to determine the necessary number of integration points for some types of finite elements at the stage of initial data reading [5].

2. Finite Element Method for Modeling Piezoceramic Structures

The mechanical and electrical phenomena as well as their coupling complicate the application of FEM to piezoceramic structures, such as actuators and other devices mentioned above. Thus, the state of each finite element nodal point can be represented by the values of the nodal displacement $\{\delta\}$ and potential $\{\varphi\}$ [1]. When describing a finite element, the displacements and potentials at any point are expressed via nodal values as:

$$\begin{cases} \{U\} = [N]\{\delta\}, \\ \{\Phi\} = [L]\{\varphi\}, \end{cases} \tag{1}$$

where $[N]$ and $[L]$ are interpolating functions in the general case, and $\{\delta\}, \{\varphi\}$ are nodal displacements and potentials, respectively.

The strain vector could be expressed in the form [18]:

$$\{\varepsilon\} = [B]\{\delta\}. \tag{2}$$

The electric field could be similarly expressed in the form [1]:

$$\{E\} = [B_E]\{\varphi\}. \tag{3}$$

Thus the equations of piezoelectric effect on the elementary volume are expressed as [16]:

$$\begin{cases} \{\sigma\} = [c^E]\{\varepsilon\} - [e]^T\{E\}, \\ \{D\} = [e]\{\varepsilon\} + [э^S]\{E\}, \end{cases} \tag{4}$$

where [c^E], [e], [$э^s$] are the matrix of stiffness for a constant electric field; the matrix of the piezoelectric constant and the matrix of dielectric constant evaluated under constant strain, respectively; {σ},{ε},{D}, {E} are the vectors of stress, strain, electric induction, and the electric field, respectively.

The analysis of a piezoelectric actuator must be performed appreciating the electric occurrence in the system. Based on FEM, each node of the element has an additional degree of freedom used for electric potentials in FEM modeling. The solution applied in the equations of motion, suitable for the actuator, can be derived from the principle of minimum potential energy by means of variation functional [4]. The basic dynamic FEM equation of motion for piezoelectric transducers that are completely covered with electrodes can be expressed as [1]:

$$\begin{cases} [M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} + [T_1]\{\phi_1\} + [T_2]\{\phi_2\} = \{F\}, \\ [T_1]^T\{u\} - [S_{11}]\{\phi_1\} - [S_{12}]\{\phi_2\} = \{Q_1\}, \\ [T_2]^T\{u\} - [S_{12}]^T\{\phi_1\} - [S_{22}]\{\phi_2\} = \{0\}, \end{cases} \tag{5}$$

where [*M*], [*K*], [*T*], [*S*], [*C*] are matrices of mass, stiffness, electro elasticity, capacity and damping, respectively; $\{u\}, \{F\}, \{Q_1\}$ are vectors of nodes of structural displacements, external mechanical forces, and charges coupled on the electrodes; $\{\phi_1\}, \{\phi_2\}$ are vectors of nodal potentials of the nodes associated with electrodes and vectors of nodal potentials calculated during numerical simulation. Mechanical and electrical boundary conditions can be applied to a piezoelectric actuator, i.e., mechanical displacement of the fixed surfaces of the actuator is equal to zero and the electric charge of piezoceramic elements that are not coupled with electrodes is equal to zero as well.

Here the structural matrices of the piezoceramic finite element with volume V are:

$$[K] = \int_V [B]^T [c^E] [B] dV, \tag{6}$$

$$[T] = \int_V [B]^T [e] [B_E] dV, \tag{7}$$

$$[S] = \int_V [B_E]^T [\ni^s] [B_E] dV, \tag{8}$$

$$[M] = \int_V [N]^T \gamma [N] dV, \tag{9}$$

$$[C] = \alpha [M] + \beta [K]. \tag{10}$$

The damping matrix $[C]$ is derived using the mass and stiffness matrices by assigning the constants α and β. γ is the mass density.

The procedure for obtaining the structural matrices of finite elements consists of the following steps:

1. Definition of interpolating functions, which contain inverse of the configuration matrix obtained from the interpolating polynomials.
2. Obtaining the derivatives of interpolating functions.
3. Multiplication of all the necessary sub-matrices.
4. Integration of these products into the volume of a finite element.

3. Comparison of Two Formation Methods

Usually piezoceramic actuators have a simple form, e.g., plates, rings, cylinders etc. [3, 7]. Therefore, finite elements of a simple form are used. Two popular finite elements will be presented here for modeling of piezoceramic actuators, namely, sector type elements and the triangular ones. The interpolating functions for the sector type finite element with four nodal points in polar coordinates (ρ, θ) could be expressed [1] as follows:

$$\{N\} = \frac{1}{(R_2 - R_1)(\theta_2 - \theta_1)} \begin{Bmatrix} (R_2 - \rho)(\theta_2 - \theta) \\ (\rho - R_1)(\theta_2 - \theta) \\ (\rho - R_1)(\theta - \theta)_1 \\ (R_2 - \rho)(\theta - \theta_1) \end{Bmatrix}, \tag{11}$$

where $R_1, R_2, \theta_1, \theta_2$ are sector values of radii and angle boundaries.

And the matrix $[B_i]$ for the i-th node of the element is [1]:

$$[B_i] = \begin{bmatrix} \frac{\partial N_i}{\partial \rho} & 0 \\ \frac{N_i}{\rho} & \frac{1}{\rho}\frac{\partial N_i}{\partial \theta} \\ \frac{1}{\rho}\frac{\partial N_i}{\partial \theta} & \frac{\partial N_i}{\partial \rho} - \frac{N_i}{\rho} \end{bmatrix}, i = 1,4. \tag{12}$$

Analogously, the matrix $[B_{Ei}]$ for the i-th node has the same form [16]:

$$\left[B_{Ei}\right]=\begin{bmatrix}\dfrac{\partial N_i}{\partial \rho}\\ \dfrac{1}{\rho}\dfrac{\partial N_i}{\partial \theta}\end{bmatrix}, i=1,4. \tag{13}$$

Thus, the expression for the stiffness matrix of the sector type finite element becomes:

$$\left[K\right]=h\int_{\theta_1}^{\theta_2}\int_{R_1}^{R_2}\left[B\right]^T\left[c^E\right]\left[B\right]\rho\, d\rho d\theta\,, \tag{14}$$

where h is the thickness of the element.

Analogously, the formulas (7-10) of the other matrices have the same structure.

The structural matrices of the sector finite element were derived using the computer algebra system VIBRAN, which is a FORTRAN preprocessor for analytical perturbation with polynomials, rational functions and trigonometric series [3]. A special VIBRAN procedure can generate an optimized FORTRAN code from the obtained analytical expressions [2], which can be directly used in the programs for numerical analysis. The computer algebra technique is more convenient for this problem than numerical, because the polynomials under integrals have negative power order -1 for the sector finite element in the polar coordinates. Thus, numerical integration is not very useful. Table 1 illustrates the number of floating point products necessary to form the structural matrices of the piezoceramic sector finite element. For the sake of clarity, let us consider only three matrices: that of mass, stiffness, electro elasticity, because their formation is more expensive.

Table 1. The number of floating point products to format the sector type element matrices

Matrix	Numerical integration	Computer Algebra
K	1296	820
T	972	634
M	2792	926
Total:	5060	2380

The other geometrically suitable finite element is triangle, which allows modeling of various surfaces [18]. The first member of a triangular membrane or a bending element family is the element with three nodes and interpolating polynomial of the second order. Also, a triangular finite element is suitable to determine the optimal electrode configuration determination of piezoceramic actuators [4]. The interpolating functions for a triangular finite element are expressed in the area of L-coordinates [5,18]:

$$\left\{N_i\right\}=\begin{Bmatrix}L_1+L_1^2L_2+L_1^2L_3-L_1L_2^2-L_1L_3^2\\ b_2\left(L_3L_1^2+0.5L_1L_2L_3\right)-b_3\left(L_1^2L_2+0.5L_1L_2L_3\right)\\ c_2\left(L_3L_1^2+0.5L_1L_2L_3\right)-c_3\left(L_1^2L_2+0.5L_1L_2L_3\right)\end{Bmatrix}, \tag{15}$$

$i=1,2,3,$

where b_i, c_i are differences of nodal coordinates, with a cyclic permutation of indices $1\to2\to3\to1$ for the remaining coefficients [5]. The matrix $\left[B_i\right]$ of the triangular element for the i-th node is [17]:

$$[B_i] = \begin{bmatrix} -\frac{\partial^2}{\partial x^2} N_i \\ -\frac{\partial^2}{\partial^2 y} N_i \\ \frac{\partial^2}{\partial x \partial y} N_i \end{bmatrix}, \tag{16}$$

$i = 1,2,3.$

Thus, the expression for the stiffness matrix of the triangular finite element becomes:

$$[K] = h \int_0^1 \int_0^{1-L_1} [B]^T \left[c^E\right] [B] \, dL_2 dL_1 , \tag{17}$$

where h is the thickness of the element.

Analogously, formulas (7-10) of the others matrices have the same structure. Integration of the products (17) in algebraic form into the area of finite element in L-coordinates according to [18] could be expressed as:

$$\int_\Delta L_1^n L_2^m L_3^k d\Delta = \frac{n!m!k!}{(n+m+k+2)!} 2\Delta , \tag{18}$$

where 2Δ is a triangle area.

Table 2 illustrates the number of floating point products necessary for the formation of structural matrices of the piezoceramic triangular finite element.

Table 2. Number of floating point products for the formation of triangular element matrices

Matrix	Numerical integration	Computer Algebra
K	2430	1204
T	1660	946
M	3240	794
Total:	7330	2944

4. Conclusions

The proposed analytical formation of the structural matrices of the piezoceramic finite elements reduces the number of floating point operations more than twice, particularly for the mass matrix of the elements. The highest economy is achieved during the integration procedure, because, after integration, the expressions of matrix elements become shorter, due to disappearance of variables. The distributed mass, electro elasticity, damping matrices always yield relatively simple final analytical expressions for complex finite elements. Due to the fact that the integral expressions to be integrated in these cases have the power higher at least by twice than the ordinary stiffness matrix, the use of computer algebra delivers a particularly remarkable economy of the required computer time as compared with the numerical integration technique. It should also be emphasized that the reduction in computational time is also accompanies by an increased accuracy, which is the most important gain achieved in many cases.

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