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Post-elastic force-displacement dependence of bent and compressed column

V. Kargaudas*, N. Adamukaitis**
**Kaunas University of Technology, Studentų 48, 51367 Kaunas, Lithuania, E-mail: vytautas.kargaudas@ktu.lt*
***Kaunas University of Technology, Studentų 48, 51367 Kaunas, Lithuania, E-mail: nerijus.adamukaitis@stud.ktu.lt*

1. Introduction

Solid mechanics, which includes the theories of elasticity and plasticity, is a broad discipline, with experimental, theoretical, and computational aspects [1]. The theory of yield plasticity does not fully satisfy experimental results. Zilinskaitė and Ziliukas in [2] presented a theory of general relation between stresses and deformation in elastic-plastic bodies and indicated the importance of inner material processes. A mathematical model for two-layer axially loaded cylindrical bars is presented by Partaukas, Bereisis [3]. The dynamic overloading and influence of kinetics on steel fracture and yield is discussed by Chausov, Pylypenko [4]. Dependence of column deformations on bending and axial forces is nonlinear if yielding stresses are attained in some cross-sections of the column. The approximate plastic-hinge approach for steel frames is presented by Powell, Chen [5], Gong [6, 7]. A method for elasto plastic large-deflection analysis with plastic hinges at midspan and two ends is proposed by Chen and Chan [8]. The empirical approximation dependences are depicted by Xu et al [9]. All these graphs are similar one to another and display only principle characteristics of real dependence.

In this paper the dependence of a column rotation and axial displacement on axial force N and bending moment M is investigated. The rotation and transverse lateral deflection, perpendicular to the column, is examined in [10]. The strain in the column is deduced in a similar way as the curvature and then longitudinal displacement is calculated by integrating with respect to longitudinal coordinate z.

An elastic-perfectly plastic stress σ dependence on strain ε is assumed (continues lines Fig. 1). Displacements of the column, deduced from this assumption, will approximately present the reality for the first loading of N and M. These calculations have to be corrected for an unloading or reloading of the column: a residual-stress distribution ([1], p. 236) and some other factors should be taken into account. The linear hardening of the mild steel (the dotted $\sigma-\varepsilon$ line for $\sigma>\sigma_Y$ in Fig. 1) more adequately depicts the real stress-strain relation, but in this case analytic approximation of the elastic and the inelastic $(\sigma>\sigma_Y)$ stress domains in every cross-section of the column is a complicated problem. The concepts of the ultimate stress distribution and the plastic hinge lose their meaning also.

2. Curvature and axial strain

Stresses in cross-section of a column can attain yielding value in both sides or only one side of the cross-section when the column is compressed and bended (Fig. 1). The cross-section is assumed to have two symmetry axis *x, y*, while width δ_0 of the flange with respect to width at the web $2h$ is neglected [11]. The influence of shear and buckling are neglected also.

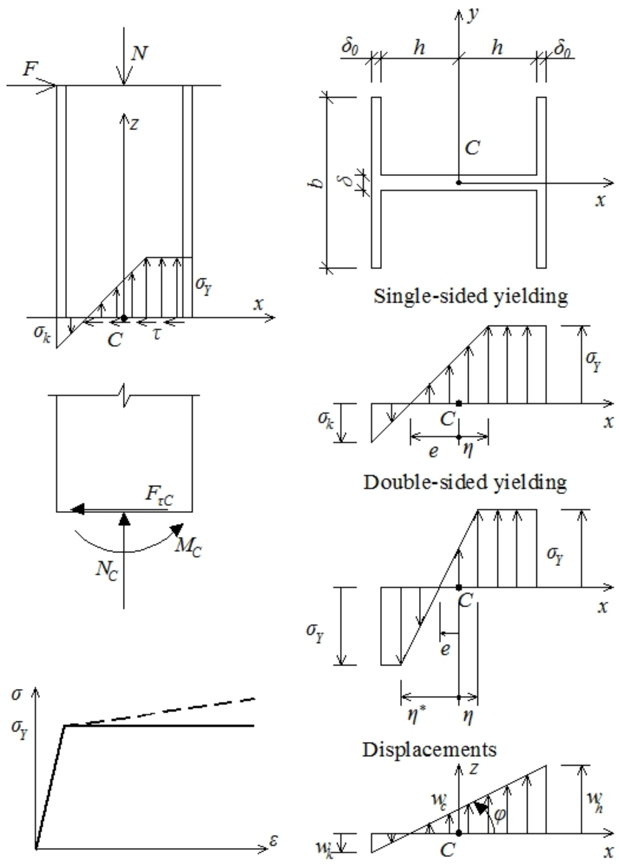

Fig. 1 Column cross-section, stresses and deflections

Deflections of cross-section $w(x)=w_c+\varphi x$, where w_c is displacement of the centroid C, φ is angle of cross-sections rotation. Strain of the fibre $\varepsilon=\dfrac{dw}{dz}=\varepsilon_c+x\dfrac{d\varphi}{dz}$ and stresses

$$\begin{cases}\sigma=\sigma_e=E\varepsilon=E\varepsilon_c+E\dfrac{d\varphi}{dz}x, & y\in A_e,\\ \sigma=\sigma_Y, & y\in A_Y\end{cases} \quad (1)$$

where A_e is domain where stresses $\sigma = \sigma_e < \sigma_Y$, A_Y is yield domain of the cross-section.

After integration of σ from Eq. (1) over the whole cross-section the resultant force N is determined, after the integration of product $\sigma \cdot x$ over the whole cross-section moment M is deduced. If equality $\varphi = \dfrac{du}{dx}$ is applied, the curvature $\dfrac{d^2u}{dz^2}$ and strain ε_c can be presented in two equations, and then

$$\begin{cases} E\left(A_e \dfrac{dw}{dz} + S_{ce}\dfrac{d^2u}{dz^2}\right) = N_c - \sigma_Y A_{YM} \\ E\left(S_{ce}\dfrac{dw}{dz} + I_{ce}\dfrac{d^2u}{dz^2}\right) = M_c - \sigma_Y S_{YM} \end{cases} \tag{2}$$

where S_{ce} is statical moment of the area A_e, I_{ce} is moment of inertia of the same area with respect to y axis. The modified area $A_{YM} = A_{Y+} - A_{Y-}$, modified statical moment $S_{YM} = S_{Y+} - S_{Y-}$ where A_{Y+}, S_{Y+} correspond the compression yield area, A_{Y-}, S_{Y-} the tension yield area. When cross-section is in single-sided yield region then $A_{Y-} = S_{Y-} = 0$ and equations can be simplified [10].

If the web area $A_1 = \delta_0 b$ and the whole area of cross-section $A = 2A_1 + 2\delta h$, then the shape coefficient is $q = 2A_1/A$. If dimensionless parameters $\alpha = \dfrac{N_c}{A\sigma_Y}$, $\beta = \dfrac{M_c}{Ah\sigma_Y}$, $K_0 = \dfrac{A_e}{A}$, $K_2 = \dfrac{S_{ce}}{Ah}$, $K_1 = \dfrac{I_{ce}}{Ah^2}$, $K_0^* = \dfrac{A_{YM}}{A}$, $K_2^* = \dfrac{S_{YM}}{Ah}$ are applied, then solution of the equations (2) can be presented

$$\begin{cases} \left(K_0K_1 - K_2^2\right)\dfrac{E}{\sigma_Y}\dfrac{dw}{dz} = \left(\alpha - K_0^*\right)K_1 - \left(\beta - K_2^*\right)K_2 \\ \left(K_0K_1 - K_2^2\right)\dfrac{Eh}{\sigma_Y}\dfrac{d^2u}{dz^2} = \left(\beta - K_2^*\right)K_0 - \left(\alpha - K_0^*\right)K_2 \end{cases} \tag{3}$$

When the area of plasticity $A_Y \to 0$ then $S_{YM} \to 0$ and $S_{ce} \to 0$, therefore Eq. (2) approaches the classical equations

$$EI_c\frac{d^2u}{dz^2} = M_c\,,\ AE\varepsilon_c = N_c \tag{4}$$

The dimensionless parameters for elastic deformations $K_0 = 1$, $K_2 = K_2^* = K_0^* = 0$, but $K_1 = (1+2q)/3$, consequently Eq.(3) are $\dfrac{E}{\sigma_Y}\dfrac{dw}{dz} = \alpha$, $\dfrac{Eh}{\sigma_Y}\dfrac{d^2u}{dz^2} = \dfrac{3\beta}{1+2q}$. If moment in any cross-section of the column $M_c = M_0(1-\xi)$, where the maximal moment $M_0 = FH$, then $\beta = \beta_0(1-\xi)$, $\xi = z/H$, $\beta_0 = M_0/Ah\sigma_Y$, where H is height of the column.

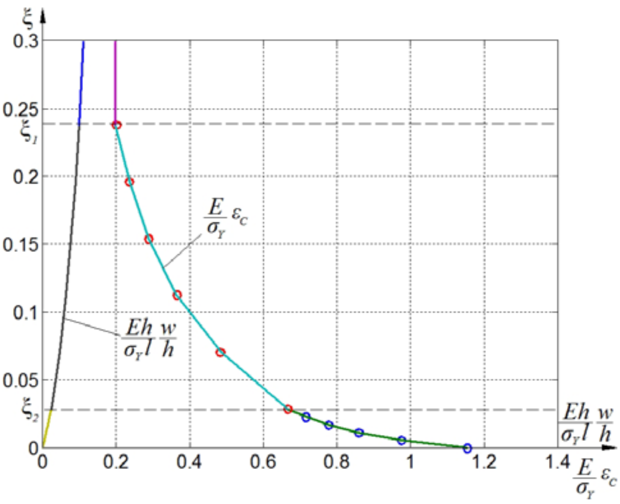

Fig. 2 Dependence of $\dfrac{E}{\sigma_Y}\varepsilon_c = \dfrac{E}{\sigma_Y}\dfrac{dw}{dz}$ and $\dfrac{Eh}{\sigma_Y H}\dfrac{w}{h}$ on the dimensionless coordinate ξ. The interval $[0, \xi_2]$ corresponds to the double-sided yield region, $[\xi_2, \xi_1]$ – to the single sided yield region. Parameters $q = 0.5$, $\alpha = 0.2$, $\beta_0 = 0.7$

Displacements of the highest point $w(H)$, $u(H)$ can be worked out integrating Eq. (2) or Eq.(3). In Fig. 2. the dependence of $\varepsilon_c = \dfrac{dw}{dz}$ on dimensionless coordinate $\xi = z/H$ is depicted. In elastic deformation region the strain $\varepsilon_c = const$, but when yield stresses are reached some polynomial approximations are determined and integrated with respect to ξ (Fig. 2).

The axial $w(z)$ and lateral $u(z)$ displacements of a column with the plastic deformations can be compared with displacements $w_e(z)$ and $u_e(z)$ of the same column and the same forces N, F applied, but yield stresses assumed $\sigma_Y \to \infty$. The displacement ratios for the highest point $w(H)/w_0(H)$ show influence of plasticity and are equal identically to unity in the elastic state regions (Fig.3.)

If plastic deformations are realized $w(H) > w_e(H)$, $u(H) > u_e(H)$ both axial and lateral displacements depend on forces N and F.

If displacements of the highest column point $w(H)$, $u(H)$ are compared with the same constant length value, independent on N and F (it can be h for example), dependences in elastic and plastic regions are different. Integration of the classical equations (4) gives the dependence of axial displacement on dimensionless axial force α and dependence of lateral displacement on dimensionless bending moment β_0

$$\frac{Eh}{\sigma_Y H}\frac{w(H)}{h} = \alpha\,,\ \frac{Eh^2}{\sigma_Y H^2}\frac{u(H)}{h} = \frac{\beta_0}{1+2q}$$

In the plastic region $w(H)$ depends on N and F, and $u(H)$ also depends on N and F (Fig. 4).

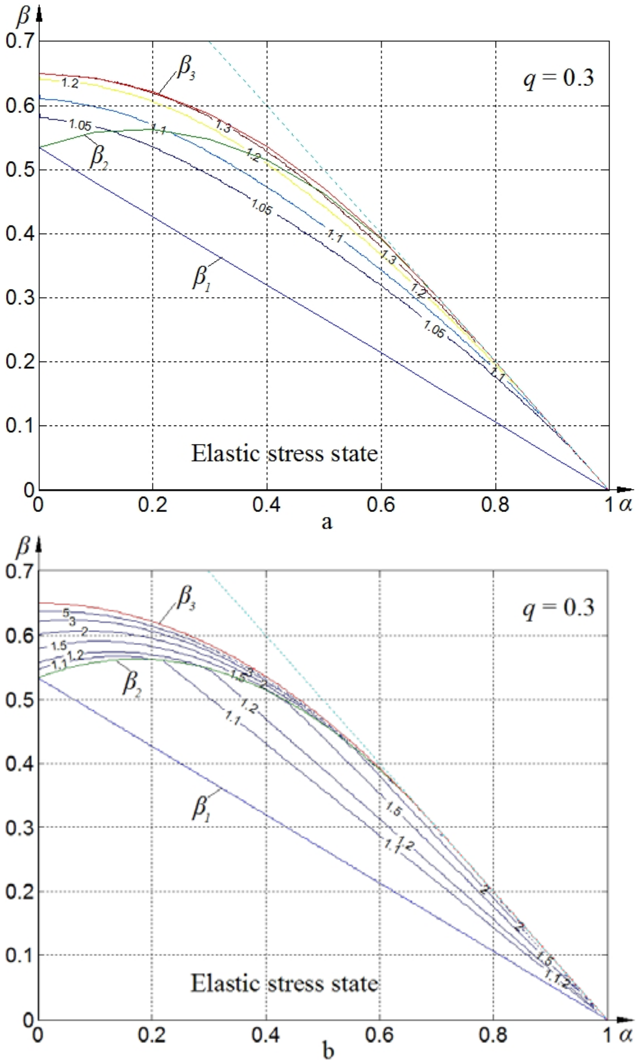

Fig. 3 Dependence of displacement ratio a - $w(H)/w_e(H)$, b - $u(H)/u_e(H)$ on axial force $\alpha = N/N_Y$ and bending moment $\beta = M/N_Y h$ when the shape factor $q = 0.3$

In Figs. 2 - 4 lines β_1 present the border of elastic state region, β_3 is plastic hinge, β_2 separates the single-sided and double-sided yield regions.

The lines of equal axial deformation $w(H) = const$ are parallel in elastic stress region (Fig. 4, a), but distances between these lines are decreasing in the plastic region when dimensionless moment $\beta = const$. Dependence of $w(H)$ on moment β when $\alpha = const$ also can be observed in the plastic region. Decreasing of distances between the lines $w(H) = const$ is the evidence of increasing of the deflections when axial force N on bending force F or both forces increase. The same conclusions can be made about the dependence of lateral deflection $u(H)$ in plastic region (Fig. 4, b). Naturally the lines in elastic region are parallel to the horizontal α axis, but in the plastic region the dependence on F and N also can be observed. The parameter of shape q influences on the distances between the lines $w(H) = const$,

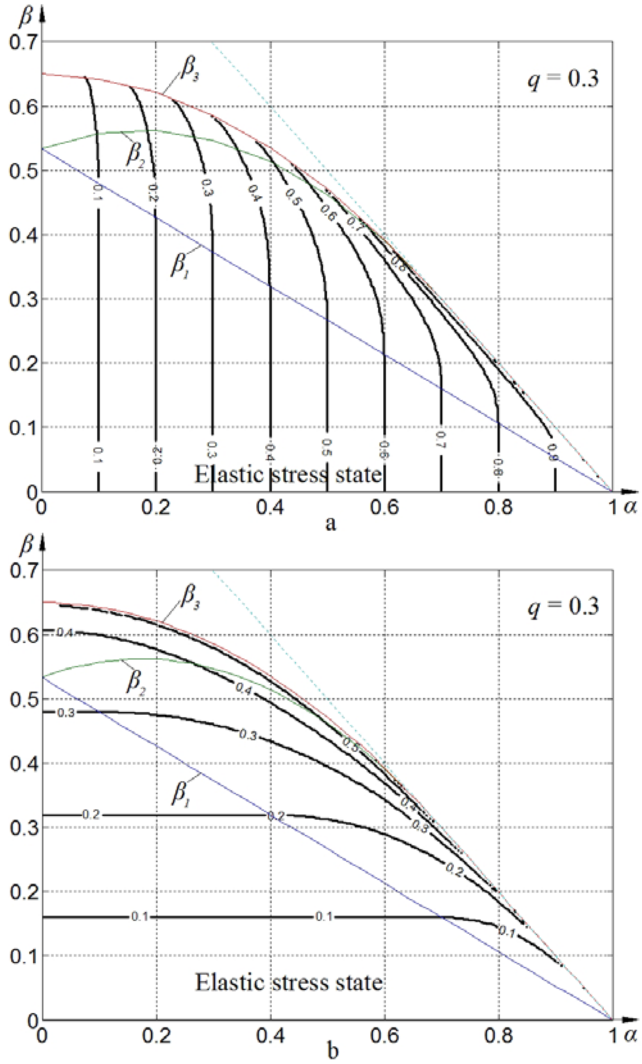

Fig. 4 Dependence of a - $\dfrac{Eh}{\sigma_Y H}\dfrac{w(H)}{h}$; b - $\dfrac{Eh^2}{\sigma_Y H^2}\dfrac{u(H)}{h}$ on axial force $\alpha = N/N_Y$ and bending moment $\beta = M/N_Y h$ when the shape factor $q = 0.3$

$u(H) = const$, but has no fundamental influence on pattern of the dependences.

3. Complex dependence of axial and transverse displacements

When plastic deformations take place axial and transverse displacements of the highest point of the column depend not only on axial force N and transverse force F correspondingly, but axial displacement $w(H)$ also depends on transverse force F and transverse displacement $u(H)$ depends on axial force N. Dependence of the plastic curvature ϕ_p on bending moment M is modeled by the equation [12]

$$\frac{d\phi_p}{dM} = \frac{c}{M_p - M_Y}\left(\frac{M - M_Y}{M_p - M}\right)^n$$

The plastic strain ε_p is defined by the equation

$$\frac{d\varepsilon_p}{dN} = \frac{c_p}{N_p - N_Y}\left(\frac{N - N_Y}{N_p - N}\right)^{n_p}$$

Some dependences of curvature on both moment and axial force are presented by Gong [5, 6], Xu et all [8]. In all these equation N_Y, M_Y are initial yield force and moment, N_p, M_p are full yield force and moment. These values correspond to the dimensionless parameters β_1, β_3 (Fig. 3).

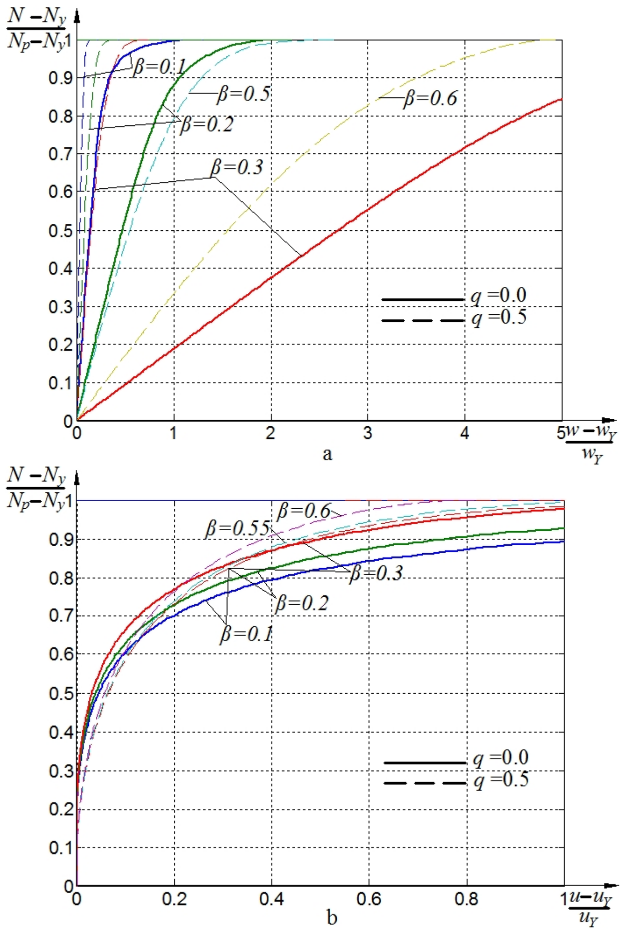

Fig. 5 Dependence curves of a - $(w - w_Y)/w_Y$ and b - $(u - u_Y)/u_Y$ on $(N - N_Y)/(N_p - N_Y)$ when $M = const$

Generally the dependence of axial deflection w on axial force N and the dependence of transverse deflection u on moment $M = FH$ are considerably more expressed than the dependence of w on moment M or of u on axial force N. The dependences of w on N and u on M in the plastic region are in some sense an extension of the linear dependences Eq. (4) and can be referred to as direct, while the dependence of N on u, or M on w as complementary.

All lines of the direct dependence display larger deflections for larger parameter β (dependence of N on w) or parameter α (dependence of M on u). The complementary dependence lines in Fig. 5 present inverse dependence for parameter β while the dependences of moment M on w are more complicated (Fig. 6).

When $q = 0.5$ and $\alpha \geq 0.5$ the direct dependence lines do not depend on α, but the lines depend highly on α if $0 \leq \alpha \leq 0.5$ (Fig. 6).

When $q = 0.5$ and $0 \leq \beta \leq 0.5$ the complementary dependence lines do not depend on β, but the lines depend highly on β if $\beta \geq 0.5$ (Fig. 5).

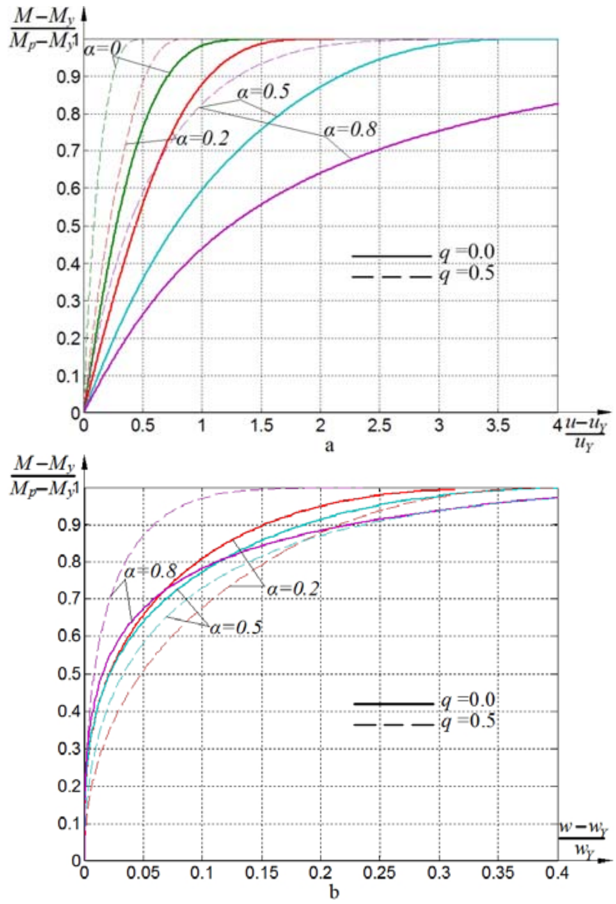

Fig. 6 Dependence curves of a - $(u - u_Y)/u_Y$ and b - $(w - w_Y)/w_Y$ on $(M - M_Y)/(M_p - M_Y)$ when $N = const$

The axial internal force N of a column is a constant along the length of the column, but bending moment M is not constant. These dependences and system of Eq. (3) suggest that complex dependences $N = N(w,u)$, $M = M(w,u)$ develop.

4. Conclusions

1. Both transverse u and axial w displacements of a column increase infinitely when the values of axial force N and bending moment M approach the limit line β_3, that is, the plastic hinge.

2. Dependence of the axial deflection w on axial force N and transverse deflection u on transverse force F is linear in the elastic deformation state and remains dominant, but not linear, in the elasto-plastic region. These dependences can be referred to as direct.

3. The complementary dependences of w on F and u on N are in general agreement with the direct de-

pendences, but complementary dependence lines have many differences from the direct dependence lines when variety of the dominant force ($N = const$ or $F = const$) is examined.

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LENKIAMŲ IR GNIUŽDOMŲ KOLONŲ JĖGŲ IR POSLINKIŲ PRIKLAUSOMYBĖ ESANT PLASTIŠKOMS DEFORMACIJOMS

R e z i u m ė

Nustatyta kolonos kreivio ir ašinės deformacijos priklausomybė nuo ašinės jėgos ir lenkimo momento, kada kai kuriuose skerspjūviuose yra plastinių deformacijų. Skaičiuojami išilginiai kolonos taškų poslinkiai. Dėl plastinių deformacijų išilginiai poslinkiai priklauso ne tik nuo ašinės jėgos, bet ir nuo lenkimo momento. Išilginės deformacijos priklausomybė nuo ašinės jėgos ir skersinės deformacijos priklausomybė nuo lenkimo jėgos yra dominantinės elastinėje-plastinėje srityje, bet atsirandanti papildoma išilginės deformacijos priklausomybė nuo lenkimo jėgos ir skersinės deformacijos priklausomybė nuo ašinės jėgos taip pat tiriama šiame straipsnyje. Šios priklausomybių linijos kinta keičiantis kolonos skerspjūvio formos parametrui.

V. Kargaudas, N. Adamukaitis

POST-ELASTIC FORCE-DISPLACEMENT DEPENDENCE OF BENT AND COMPRESSED COLUMN

S u m m a r y

Dependences of a column rotation and axial strain on axial force and bending moment are deduced when plastic strain are in some cross-sections. When post-elastic deformation are in the column the axial displacements depend not only on the axial force but also on the bending moment. Dependences of axial deformation on axial force and transverse deformation on bending force are dominant in the elasto-plastic region, but the complementary dependence of the axial deformation on the bending force and the dependence of the transverse deformations on the axial make themselves evident and are investigated in this paper also. These lines of dependences vary with the cross-section shape factor.

В. Каргаудас, Н. Адамукайтис

ЗАВИСИМОСТЬ МЕЖДУ СИЛОЙ И СМЕЩЕНИЕМ ДЛЯ ИЗГИБАЕМЫХ И СЖИМАЕМЫХ КОЛОНН ПРИ ПЛАСТИЧЕСКИХ ДЕФОРМАЦИЯХ

Р е з ю м е

Определена зависимость кривизны и продольной относительной деформации колонны от продольной силы и изгибающего момента, когда в некоторых сечениях колонны возникают пластические деформации. Вычисляются продольные смещения точек колонны. Из-за пластических деформаций продольные смещения зависят не толкьо от сжимающей силы, но и от изгибающего момента. Зависимость продольной деформации от продольной силы и поперечной деформации от изгибающей силы является преобладающей в упруго-пластической зоне, но возникающие дополнительные зависимости продольной деформации от изгибающей силы и поперечной деформации от продольной силы также исследуются в этой статье. Линии этих зависимостей изменяются при изменении параметра формы поперечного сечения.

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