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# Real Time Analysis of Accelerometer Pair's Observations Based on Maximum Relative Entropy Optimization Satisfying Model Constraints

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### Introduction

Maximum relative Entropy (*MrE*) optimization principles were studied in previous paper [1] where convex optimization was performed incorporating model constraints resulting from one axis differential drive robot's constraints such as the distance between wheels. *MrE* Lagrangian was constructed for just one discrete observation, so it had several things still to be considered.

First, uniform prior was selected when applying Bayesian filter for recorded data. While principle of entropy maximization states that the distribution closest to a priori knowledge has to be selected.

Second, resulting entropy multiple integral did not have analytical solution of its antiderivative. When solving this optimization problem had lead to the use of numerical integration.

Third, observation data arrived in high volumes so having numerical iterations influenced method's performance as soon as the number of observation data became large.

### Updating with observation data constraints

What is a priori knowledge when having observation data and knowing the model on how different observation channels relate to each other? The answer is inferred from the knowledge collected so far. First thing which is known states that discrete observation's estimate must have its expectation at the exact value observed, i.e. its mean has to fall right at the observed value, and it is not being able to become "unobserved" (see work by A.Giffin and A.Caticha for wider discussion regarding this [2]). Dirac delta function was used in previous work, but expectation constraint is used for simplicity here. So we already have four constraints

$$\int_{-\infty}^{+\infty} a_x P(a_x) \mathrm{d}a_x = a_{x,obs} = c_{ax}, \quad (1)$$

$$\int_{-\infty}^{+\infty} a_y P(a_y) \mathrm{d}a_y = a_{y,obs} = c_{ay}, \quad (2)$$

$$\int_{-\infty}^{+\infty} b_x P(b_x) \mathrm{d}b_x = b_{x,obs} = c_{bx}, \quad (3)$$

$$\int_{-\infty}^{+\infty} b_y P(b_y) \mathrm{d}b_y = b_{y,obs} = c_{by}, \quad (4)$$

where a pair of two-axis accelerometers (*z* axis is not taken into account in this work) is represented by letters *a* and *b*. Thus $a_{x,obs}, a_{y,obs}$ are observations of *x*, *y* axis of accelerometer *a*. Consequently, $b_{x,obs}, b_{y,obs}$ are observations of *x*, *y* axis of accelerometer *b*. The very first entropy optimization is a simultaneous optimization taking into account the mathematical model between all four observation channels.

A priori distribution is calculated using the measure of entropy for the continuous-variate Probability Density Function (*PDF*):

$$S[P] = -\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \begin{pmatrix} P(a_x, a_y, b_x, b_y)\cdot \\ \mathrm{Ln}(P(a_x, a_y, b_x, b_y)) \end{pmatrix} \mathrm{d}a_x \mathrm{d}a_y \mathrm{d}b_x \mathrm{d}b_y \cdot \quad (5)$$

here $P(a_x), P(a_y), P(b_x), P(b_y)$ as from formulas (1) – (4) are marginal *PDF*s respectively as follows

$$P_{ax} = P(a_x) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P(a_x, a_y, b_x, b_y) \mathrm{d}a_y \mathrm{d}b_x \mathrm{d}b_y, \quad (6)$$

$$P_{ay} = P(a_y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P(a_x, a_y, b_x, b_y) \mathrm{d}a_x \mathrm{d}b_x \mathrm{d}b_y, \quad (7)$$

$$P_{bx} = P(b_x) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P(a_x, a_y, b_x, b_y) \mathrm{d}a_x \mathrm{d}a_y \mathrm{d}b_y, \quad (8)$$

$$P_{by} = P(b_y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P(a_x, a_y, b_x, b_y) \mathrm{d}a_x \mathrm{d}a_y \mathrm{d}b_x \cdot \quad (9)$$

Then normal distribution's normalization constraint is again incorporated into final Lagrangian as

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P(a_x, a_y, b_x, b_y)\,\mathrm{d}a_x \mathrm{d}a_y \mathrm{d}b_x \mathrm{d}b_y = 1 \cdot \quad (10)$$

All constraints have been enumerated except those of mathematical model. The mathematical model is necessary to infer the shape of *PDF* for every channel. In other words, if we have a mathematical model which shows how all four channels relate we can infer which current observation of all four channels fits the model better and which fits worse. Second thing, which has to be considered, is that so far constraints contained the updating of the first probabilistic moment, i.e. the expectation. The second probabilistic moment can be taken from the mathematical model, so it has to be selected with caution.

**Updating with model and observation data constraints**

An experiment was performed with two accelerometers mounted on a rigid body (robot). The main aim was to infer both accelerometers' *x* axis zero bias from dynamic real time observation data. Axis *y* had a very small zero bias so it was neglected in zero bias. So it was decided to perform an autocalibration experiment. During it rigid body was being rotated around static rotation center where all geometrical distances were known (see figure (1)).

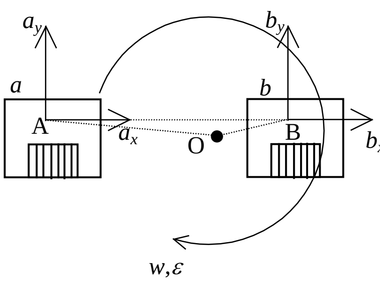


**Fig. 1**. Autocalibration experiment with two accelerometers mounted on rigid body (robot) and rotating about geometrically known rotation center

In other words, instant center of rotation during the whole autocalibration experiment was kept constant and distances *OA*, *OB* and *AB* geometrical layout were known. Moreover, angular velocity *w* was being tried to be kept as constant as possible, so it was known that angular acceleration fluctuations were small and there was an initial honest knowledge that *w* did not exceed values $w_{max}$ and $w_{min}$. Angular velocity and acceleration were derived from rigid body kinematics relations. One can derive two main relationships between all four observation channels

$$a_x^2 + a_y^2 = k_1\left(b_x^2 + b_y^2\right), \quad (11)$$

$$a_x b_x + a_y b_y = k_2\left(b_x^2 + b_y^2\right). \quad (12)$$

And this gives two more constraints which represented model constraints, i.e.

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left(a_x^2 + a_y^2\right) P(a_x, a_y, b_x, b_y)\,\mathrm{d}a_x \mathrm{d}a_y \mathrm{d}b_x \mathrm{d}b_y = $$
$$= k_1 \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left(b_x^2 + b_y^2\right) P(a_x, a_y, b_x, b_y)\,\mathrm{d}a_x \mathrm{d}a_y \mathrm{d}b_x \mathrm{d}b_y \quad (13)$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left(a_x b_x + a_y b_y\right) P(a_x, a_y, b_x, b_y)\,\mathrm{d}a_x \mathrm{d}a_y \mathrm{d}b_x \mathrm{d}b_y = $$
$$k_2 \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left(b_x^2 + b_y^2\right) P(a_x, a_y, b_x, b_y)\,\mathrm{d}a_x \mathrm{d}a_y \mathrm{d}b_x \mathrm{d}b_y \cdot \quad (14)$$

Constructing Lagrangian using (5) and the constraints (1) – (4), (10), (13), (14) require the definition of Lagrange multipliers $k_{ax}$, $k_{ay}$, $k_{bx}$, $k_{by}$ for constraints (1) – (4), $\beta$ for constraint (13) and $\rho$ for constraint (14) respectively.
Then it can be proved that the marginal *PDFs* are

$$P_{ax} = \frac{2\pi^{\frac{3}{2}} e^{\frac{\beta\left(k_{by}^2 - k_1 k_{ay}^2\right) - \rho\left(k_2 k_{ay}^2 - k_{ay} k_{by}\right)}{c_1} + \frac{k_{bx}^2 + 2a_x\left(2\beta k_1 k_{ax} + \rho\left(2k_2 k_{ax} + k_{bx}\right)\right) + a_x^2 c_1}{\beta k_1 + k_2\rho}}}{c_{norm}\sqrt{c_1\left(-\beta k_1 - k_2\rho\right)}}, \quad (15)$$

$$P_{ay} = \frac{2\pi^{\frac{3}{2}} e^{\frac{\beta\left(k_{bx}^2 - k_1 k_{ax}^2\right) - \rho\left(k_2 k_{ax}^2 - k_{ax} k_{bx}\right)}{c_1} + \frac{k_{by}^2 + 2a_y\left(2\beta k_1 k_{ay} + \rho\left(2k_2 k_{ay} + k_{by}\right)\right) + a_y^2 c_1}{\beta k_1 + k_2\rho}}}{c_{norm}\sqrt{c_1\left(-\beta k_1 - k_2\rho\right)}}, \quad (16)$$

$$P_{bx} = \frac{2\pi^{\frac{3}{2}} e^{-\beta b_x^2 k_1 + b_x\left(k_{bx} - b_x k_2\rho\right) - \frac{\left(k_{ax} + b_x\rho\right)}{4\beta} + \frac{\beta\left(k_{by}^2 - k_1 k_{ay}^2\right) - \rho\left(k_2 k_{ay}^2 - k_{ay} k_{by}\right)}{c_1}}}{c_{norm}\sqrt{c_1\beta}}, \quad (17)$$

$$P_{by} = \frac{2\pi^{\frac{3}{2}} e^{-\beta b_y^2 k_1 + b_y\left(k_{by} - b_y k_2\rho\right) - \frac{\left(k_{ay} + b_y\rho\right)}{4\beta} + \frac{\beta\left(k_{bx}^2 - k_1 k_{ax}^2\right) - \rho\left(k_2 k_{ax}^2 - k_{ax} k_{bx}\right)}{c_1}}}{c_{norm}\sqrt{c_1\beta}}, \quad (18)$$

where normalization constant $c_{norm,}$ Lagrange multipliers, and constant $c_1$ are

$$c_1 = 4\beta^2 k_1 + 4\beta k_2\rho + \rho^2, \quad (19)$$

$$c_a = c_{ax}^2 + c_{ay}^2, \quad (20)$$

$$c_b = c_{bx}^2 + c_{by}^2, \quad (21)$$

$$c_2 = c_a^2 + 2c_{ax}^2 c_{bx}^2 k_1 - 2c_{ay}^2 c_{bx}^2 k_1 + 8c_{ax}c_{ay}c_{bx}c_{by}k_1 - 2c_{ax}^2 c_{by}^2 k_1 + $$
$$2c_{ay}^2 c_{by}^2 k_1 + c_b^2 k_1^2 - 4k_2\left(c_{ax}c_{bx} + c_{ay}c_{by}\right)\left(c_a + k_1 c_b\right) + 4c_a c_b k_2^2, \quad (22)$$

$$c_{norm} = \frac{4\pi^2 e^{\frac{\beta\left(k_{bx}^2 + k_{by}^2 - k_1\left(k_{ax}^2 + k_{ay}^2\right)\right) - \rho\left(k_2\left(k_{ax}^2 + k_{ay}^2\right) - k_{ax}k_{bx} - k_{ay}k_{by}\right)}{c_1}}}{c_1}, \quad (23)$$

$$k_{ax} = \left(\begin{matrix} -4k_1\left(\begin{matrix} c_{ax}^3 + 2c_{ay}c_{bx}c_{by}k_1 + \\ c_{ax}k_1\left(c_{bx} - c_{by}\right)\left(c_{bx} + c_{by}\right)\end{matrix}\right) - \\ 8c_b c_{ax} k_2^2 + 4k_2\left(\begin{matrix} 2c_{ax}c_{ay}c_{by} + c_{ay}^2 c_{bx} + \\ 3c_{ax}^2 c_{bx} + c_{bx}c_b k_1 \end{matrix}\right)\end{matrix}\right) / c_2 \quad (24)$$

and Lagrange multipliers $k_{ay}$, $k_{bx}$, $k_{by}$ are calculated similarly. Coefficients $k_1$ and $k_2$ depend on geometrical layout of accelerometers on the rigid body, i.e. they are calculated from *OA*, *OB* and *AB* distances.

It can be seen from formulas (15) – (24) that antiderivative function for *MrE PDF* calculation is found.

So *MrE* optimization when finding a priori *PDF* is an O(1) operation, which makes this method practical. An attentive reader would notice that

$$\arg\max_{a_x} P_{ax} = \int_{-\infty}^{+\infty} a_x P(a_x) \mathrm{d}a_x, \quad (25)$$

$$\arg\max_{a_y} P_{ax} = \int_{-\infty}^{+\infty} a_y P(a_y) \mathrm{d}a_y, \quad (26)$$

$$\arg\max_{b_x} P_{bx} = \int_{-\infty}^{+\infty} b_x P(b_x) \mathrm{d}b_x, \quad (27)$$

$$\arg\max_{b_y} P_{bx} = \int_{-\infty}^{+\infty} b_y P(b_y) \mathrm{d}b_y \quad (28)$$

and it means that Maximum Likelihood (*ML*) estimate falls at the observation reading just like expectation does, and it confirms that the approach to treat likelihood included in the a priori *PDF* was correct indeed, see work [2]. In other words,

$$P_{old}(a_x, c_{ax}) = P_{old}(c_{ax}) \cdot P_{old}(a_x \mid c_{ax}, c_{ay}, c_{bx}, c_{by}), \quad (29)$$

where model constraints (11) and (12) are updated with constraints moments.

**Updating time series distributions with model constraints**

Time series observation data can contain not only a static noise which can be filtered with regular autoregression efforts, but also a dynamic noise which could be a result of external forces like a hit to the wall when accelerometers had an instantaneous peak. Then averaging does not help even if a huge number of observation data is collected. There is one more property of accelerometers which has to be considered. Every axis has its zero bias. It can be observed by turning the axis to match the direction of Earth's gravity force, but the problem is that this zero drift is also dynamic, i.e. zero might drift because accelerometer might be aging. Aging detection and causes are not explored in this paper, but the question whether the unbiased estimator can be found for calculation of current zero bias can be answered with *yes*. Not without the help of *MrE* updating with model constraints. Assume *n* readings are observed for every accelerometer's channel, and assume that at the time moment *i* the following measurements are observed: $c_{axi}$, $c_{ayi}$, $c_{bxi}$, $c_{byi}$.

Other coefficients and variables also have their notation with index *i*. Then Lagrangian can be constructed with relative entropy formula

$$S[P, P_{\mathrm{old}}] = -\sum_{i=1}^{n} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left( P_i(a_x, a_y, b_x, b_y) \cdot \mathrm{Ln} \frac{P_i(a_x, a_y, b_x, b_y)}{P_{\mathrm{old,i}}(a_x, a_y, b_x, b_y)} \right) \mathrm{d}a_x \mathrm{d}a_y \mathrm{d}b_x \mathrm{d}b_y, \quad (30)$$

where

$$P_{\mathrm{old,i}}(a_x, a_x, a_x, a_x) = P_{\mathrm{old,i}}(a_x) P_{\mathrm{old,i}}(a_y) P_{\mathrm{old,i}}(b_x) P_{\mathrm{old,i}}(b_y), \quad (31)$$

and each multiplier is just a *PDF* taken from (15) – (18) for every channel respectively. This implies that noncommuting constraints (see work [1,2] for discussion on commutativity) will be used when comparing it to the observation data constraints. In other words, observation data plays its role for calculating *MrE* distributions for time moment *i* for each channel. And from pre-calculated Lagrange multipliers can be used for further inference.

Important question is whether first or second probabilistic moment will be used for updating with model constraints. After multiple cases analysis it was found that updating with second probabilistic moment gives little benefit when finding an analytical posteriori. And updating with first probabilistic moment would help avoid time consuming antiderivative numerical calculations. After some manipulation with (11) and (12) it is found that these constraints could be rewritten as

$$a_{xi} = b_{xi} k_2 \pm b_{yi} \sqrt{k_1 - k_2^2}, \quad (32)$$

$$a_{yi} = b_{yi} k_2 \mp b_{xi} \sqrt{k_1 - k_2^2}, \quad (33)$$

which makes them acceptable as model constraints and preserves relatively easy calculation of antiderivative function of the entropy multiple integral (30). However, with *n* observations there are *4n* unknowns and only *2n* equations were at hand so far. Then two more model constraints are used which bring more constraints on time series over time domain, i.e.

$$a_{x,i} - a_{x,i+1} = c_3 + b_{x,i} - b_{x,i+1}, \quad (34)$$

$$a_{y,i} - b_{y,i} = c_4, \quad (35)$$

where constant $c_3$ and $c_4$ are calculated using cubic spline derivations when three main inputs (in addition to rough geometrical layout distances) are taken into the account: $\Delta t$ which was a time period between two samples, $w_{min}$ and $w_{min}$ were maximum and minimum angular velocities during the whole autocalibration experiment according to a priori knowledge.

A raw *MrE* calculation has a performance of O(*n*), but the problem is that the signs in formulas (32) and (33) are not known in advance. Because of that local optimization over a certain window has to be run first. The local processing window (*LPW*) is selected to contain 6 samples, which give an asymptotic notation as $\mathrm{O}(6n\mathrm{Log}_2 n)$. Unlike particle filters methods [3], this method has the analytical representation of normalization constant. Also it shows how to extract more information from dynamic measurements (see work [4]).

After the baseline of observation data was found satisfying all time-domain model constraints, the unbiased estimator for calculating zero bias can be approximated and found from Normal distribution as

$$a_{x,bias} = \sum_{i=1}^{n} \left( a_{x,entropy,i} - c_{ax} \right) \Big/ n \approx 0.01g, \quad (36)$$

$$b_{x,bias} = \sum_{i=1}^{n} \left( b_{x,entropy,i} - c_{bx} \right) \Big/ n \approx 0.065g, \quad (37)$$

where *g* is Earth's gravity.

## Conclusions

Analytical and iterations-free antiderivative function has been derived for *MrE* optimization with model constraints. The estimated zero bias values are compared to the values observed from static bias when *x* axis is being pointed to the direction of Earth's gravity direction, and it is found to be consistent.

Multiple dependent cheap sensors can be used in inference with the same benefit which would be given by a single expensive sensor, when the sufficient number of observation data is available. Aging of a sensor can be monitored from dynamic observation data.

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Analysis of two accelerometers' observation data using maximum relative entropy principles by satisfying both time series and model constraints is presented. Convex optimization principles are used to find the maximum entropy priori probability function. And the likelihood is inferred simultaneously during a priori calculations. Performance is taken into account when developing real time analysis with iteration free optimization. As a result zero bias estimation function is derived for accelerometers' axis and it confirms the value which is manually found when observing Earth's gravity under static conditions, i.e. when accelerometer is not moving. Ill. 1, bibl. 5 (in English; abstracts in English, Russian and Lithuanian).

**Р. Урнежюс, С. Барткевичюс, С. Жебраускас. Локализация робота при динамической неопределенности // Электроника и электротехника. – Каунас: Технология, 2010. – № 4(100). – С. 43–46.**

Представлен анализ данных наблюдения двух акселерометров, используя принципы максимальной относительной энтропии. Одновременно были удовлетворены ограничения модели и дискретных данных. Чтобы найти априори вероятностной функции максимальной энтропии использовались принципы выпуклой оптимизации. В течение априорных вычислений одновременно выведена функция правдоподобия. Развивая оперативный анализ во внимание принято быстродействие и развит безитерационный метод. В результате получена функция оценки уклона нуля для оси акселерометров, и это подтверждило число, которое найдено, наблюдая тяготение Земли при статических условиях, то есть когда акселерометр не перемещался. Ил. 1 , библ. 5 (на английском языке; рефераты на английском, русском и литовском яз.).

**R. Urniežius, S. Bartkevičius, S. Žebrauskas. Akcelerometrų poros signalų analizė pagrįsta santykinės entropijos metodo optimizacija įvertinus modelio ribojimus // Elektronika ir elektrotechnika. –Kaunas: Technologija, 2010. – Nr. 4(100). – P. 43–46.**

Naudojantis maksimalios santykinės entropijos principais pateikiama dviejų akcelerometrų rodmenų analizė. Tuo pačiu metu buvo tenkinami realaus laiko duomenų ir modelio ribojimai. Maksimalios entropijos aprioriniam tikimybiniam skirstiniui rasti buvo panaudoti iškiliojo optimizavimo principai. Aprioriniais skaičiavimais taip pat buvo gauta tikėtinumo funkcija. Realaus laiko duomenų analizė atliekama įvertinant greitaveikos poreikį neiteraciniu būdu. Gauta kiekvieno akcelerometro ašies nulio poslinkio įvertinio funkcija. Jos sprendinys patvirtintas stebint šį poslinkį statinėmis sąlygomis, t. y. kai akcelerometro ašis nukreipta lygiagrečiai su Žemės laisvojo kritimo pagreičio kryptimi. Il. 1, bibl. 5 (anglų kalba; santraukos anglų, rusų ir lietuvių k.).