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Algebraic reduction of fractional systems: Analytical solutions and memory-induced soliton deformation

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ABSTRACT

This study presents a novel algebraic methodology that transforms systems of nonlinear Caputo fractional differential equations into equivalent first-order ordinary differential equations. By applying the properties of fractional power series, the proposed framework maps non-local fractional operators directly into a local integer-order domain, embedding infinite memory effects into a deterministic, time-dependent polynomial forcing term. The physical applicability of this reduction technique is investigated through the analysis of a multiplicatively coupled fractional Riccati system, specifically focusing on the deformation of solitary wave solutions induced by fractional memory. Computational experiments indicate that non-zero fractional initial conditions function as continuous shape modulators that fundamentally alter the amplitude, width, and asymptotic decay rates of solitary waves, thereby breaking the exact symmetries of standard localized behavior. Despite this memory-induced deformation and the associated loss of general integrability, novel exact analytical solutions are successfully constructed for the perturbed system. By identifying specific parameter constraints that compensate for the non-autonomous memory effects, the nonlinear ordinary differential equation system is mapped into a higher-dimensional linear space and resolved using Bessel functions. The resulting closed-form analytical solutions, represented as exact integrals, are validated against direct numerical approximation methods, confirming the accuracy and practical utility of the proposed mathematical framework for analyzing complex fractional dynamics.

1. Introduction

The transition from classical integer-order calculus to fractional calculus represents a significant advancement in the mathematical modeling of complex physical phenomena. While classical dynamical systems are inherently Markovian, depending exclusively on their current state, fractional differential equations (FDEs) utilize integro-differential operators to incorporate past states into their mathematical models, effectively giving the system a memory of its previous dynamics. Among the various formulations, the Caputo fractional derivative is widely utilized in applied mathematics because it effectively models long-range temporal correlations and power-law fading memory [1–3]. While alternative operators, such as, for instance, fractal-fractional [4] or two-scale fractal derivatives [5], offer excellent mathematical frameworks for modeling multi-scale spatial geometries and transport in highly porous

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media, the classical Caputo operator remains the standard choice for purely temporal, non-Markovian dynamics that do not require the assumption of a fractal spatial geometry. Consequently, Caputo fractional models have become essential in diverse scientific disciplines [6–8]. For instance, in epidemiology, fractional operators can be utilized to capture the transmission dynamics of complex infectious diseases, which possess intrinsic latency periods, varying incubation times, and transmission rates that depend heavily on the historical trajectory of the outbreak [9,10]. Similarly, in population biology, fractional operators are applied to population growth models to ensure that the memory and hereditary properties of a species, along with complex environmental uncertainties, are accurately reflected in its evolutionary dynamics [11,12]. In economics and finance, Caputo derivatives are used to model the time-varying memory effects inherent in consumer purchasing behavior, capital accumulation, and the long-range dependencies governing monetary policy, credit risk, and market volatility [13,14]. Furthermore, in the physical sciences, fractional operators are utilized to accurately simulate complex wave phenomena, such as non-local heat transfer mechanisms in engineered media [15] and memory-driven wave deformation in fluid dynamics [16].

Despite their profound modeling capabilities, solving systems of nonlinear FDEs remains an analytical and computational challenge. The non-local nature of fractional operators often causes classical techniques to fail, necessitating specialized approaches that are broadly categorized into analytical, semi-analytical, and numerical techniques. Analytical methods, such as the application of the integral transforms [17,18] or matrix generalizations of the Mittag-Leffler function [19] aim to provide exact closed-form solutions. While highly effective for solving linear, commensurate systems of FDEs, these techniques often become mathematically intractable when applied to complex coupled non-linear or incommensurate systems [18]. The inability to compute exact inverse transformations or scale massive matrices for chaotic dynamics and complex boundaries ultimately forces a necessary pivot toward more adaptable approximation strategies. Semi-analytical techniques combine the mathematical rigor of analytical derivations with the algorithmic flexibility of numerical computing by approximating solutions to complex, nonlinear fractional systems through continuous, differentiable infinite series expansions. Approaches like the Adomian decomposition method [20] and the homotopy analysis method [21,22] utilize specialized polynomials and topological deformations, preserving nonlinear dynamics without relying on restrictive local linearization. Furthermore, hybridizing these techniques with generalized integral transforms [23] or residual error minimization [24] enhances convergence rates and mitigates the computational bottlenecks associated with evaluating highly nonlinear coupled systems. Additionally, recent developments in residual power series techniques, particularly those hybridized with Laplace (LRPS) and Sumudu (SRPS) transforms, have demonstrated great efficiency in constructing highly accurate semi-analytical solutions for both linear and nonlinear fractional systems [25–28]. Despite their capability to generate highly accurate closed-form polynomial expressions, the fundamental limitation of all semi-analytical methods lies in their strictly bounded radius of convergence [29]. As these infinite series naturally diverge across extended temporal or spatial horizons, researchers are inevitably forced to transition to discrete numerical methodologies for long-term simulations. Numerical techniques discretize continuous spatial and temporal domains to iteratively approximate fractional states across large-scale, chaotic, or geometrically irregular systems. Discretization frameworks such as predictor–corrector algorithms [30] and spectral collocation methods [31] utilize advanced interpolation polynomials and global orthogonal polynomial basis functions to maintain rigorous stability bounds and exponentially reduce dispersion errors. Furthermore, to mathematically resolve the non-smooth singularities inherent to Caputo fractional derivatives at the initial time step, modern finite difference formulations implement highly graded meshes that densely concentrate grid points near the origin, thereby restoring optimal global convergence rates [32]. However, because these schemes do not provide exact mathematical functional expressions, their reliance on the infinite memory effect intrinsic to the Caputo derivative forces the continuous evaluation of the entire historical trajectory at each sequential step, leading to an overwhelming computational overhead.

To circumvent the limitations of these conventional methodologies – specifically, the bounded convergence radii of semi-analytical techniques and the severe computational memory overhead of direct numerical solvers for system-level dynamics – this paper proposes an analytical technique that utilizes fractional power series to map fractional operators entirely onto a local integer-order space. Furthermore, unlike traditional analytical integral transforms that frequently become mathematically intractable when applied to complex nonlinearly coupled systems, this algebraic reduction applies the generating function isomorphism to natively handle cross-coupling and nonlinearities without requiring approximations. By transitioning from the fractional domain to the integer-order domain, this approach completely avoids the integration difficulties typically associated with solving FDEs, as it only requires solving ordinary differential equations (ODEs). This reduction to ODEs opens up a far more extensive and robust set of mathematical techniques. In previous studies, it was demonstrated that utilizing the fractional power series algebra, a single, isolated Caputo fractional differential equation can be systematically transformed into an equivalent first-order ODE [33,34]. While this prior research successfully established the exact algebraic reduction for scalar fractional equations, the primary objective of the present study is to rigorously expand this methodology. Specifically, this paper aims to generalize this reduction technique from single equations to nonlinear systems of Caputo FDEs.

To demonstrate the physical applicability of the proposed technique, the methodology is applied to a multiplicatively coupled fractional Riccati system. In standard integer-order dynamics, specific parameter configurations of this system are known to admit solitary solutions [35]. Solitary waves, or solitons, are fundamental wave packets that maintain structural integrity over extensive distances due to a precise balance between nonlinearity and dispersion. This exceptional stability makes them highly applicable – from modeling energy transport in shallow waters and complex marine environments [36,37] and implementing ultra-high-speed data transmission preventing signal degradation over transoceanic distances [38] to optimizing sensor properties [39] and developing quantum computing systems [40]. Because sustaining these waves requires such a delicate balance of competing effects, they provide an excellent framework for studying the impact of non-local operators. Thus, the second objective of this paper is to investigate the physical deformation of these waves induced by fractional memory effects. As is frequently observed in broader

studies of dynamical systems, the introduction of fractional memory fundamentally alters the underlying physics. In these general contexts, fractional derivatives are known to function as non-autonomous forcing terms that break classical spatial and temporal symmetries [41], acting as continuous shape modulators that introduce intrinsic dissipative effects and alter the amplitude, steepness, and symmetry of standard wave profiles [42]. While it is anticipated that the introduction of such memory perturbations will similarly deform classical solitons and compromise the exact integrability of the Riccati system, analytical solutions may still be recovered under specific constraints. Therefore, the third objective of this study is to establish that by identifying parameter conditions that compensate for these memory effects, the memory-perturbed fractional Riccati system can still be solved exactly, yielding novel closed-form analytical solutions.

To summarize, the primary novelty of this work lies in generalizing algebraic reduction techniques from scalar equations to systems of nonlinear Caputo FDEs. Furthermore, this study explores the physical applicability of this extended framework by analyzing how fractional memory deforms known stable wave structures, and demonstrates how the proposed technique can be utilized to construct novel, exact analytical solutions.

The remainder of this paper is organized as follows. Section 2 outlines the foundational concepts of the fractional power series algebra and the Caputo fractional derivative operator. Section 3 details the proposed reduction of nonlinear Caputo fractional systems to the systems of ODEs. Section 4 investigates the impact of fractional memory on multiplicatively coupled Riccati systems, analyzing the deformation of solitary solutions and deriving exact analytical solutions. Finally, concluding remarks are presented in Section 5.

2. Preliminaries

This section outlines the mathematical concepts required for the subsequent analysis. The study focuses on the Caputo fractional derivative, which for a function f and derivative order $\alpha > 0$ is defined by the integral expression [2]:

$$ {}^{C}\mathbf{D}^{(\alpha)} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, \mathrm{d}\tau, \quad x > 0, \tag{1} $$

where $n = \lceil \alpha \rceil$ and $\Gamma(\cdot)$ is the Gamma function. Rather than relying directly on this integral operator, the analysis utilizes an algebraic framework based on fractional power series, where the Caputo derivative acts as a linear operator on the basis functions [33].

2.1. Fractional power series algebra

Let the parameter $n \in \mathbb{N}$ be fixed. The mathematical framework, referred to as the Caputo algebra, is outlined in this section. This framework allows the treatment of fractional differential equations via algebraic techniques analogous to operational calculus [33].

Definition 2.1. The fractional basis functions $w_j^{(n)}$ for $j = 0, 1, 2, \ldots$ are defined as:

$$ w_j^{(n)}(x) = \frac{x^{j/n}}{\Gamma\left(\frac{j}{n}+1\right)}. \tag{2} $$

Definition 2.2. A Caputo fractional power series f is defined as an expansion over the basis functions:

$$ f = \sum_{j=0}^{\infty} c_j w_j^{(n)}, \quad c_j \in \mathbb{C}. \tag{3} $$

The set containing all such fractional power series is denoted by ${}^{C}\mathbb{F}_n$. Addition and scalar multiplication operations within this set are defined component-wise.

Definition 2.3. Let $f = \sum_{j=0}^{\infty} c_j w_j^{(n)}$ and $g = \sum_{j=0}^{\infty} b_j w_j^{(n)}$ be elements of ${}^{C}\mathbb{F}_n$. Their product is defined in the generalized Cauchy sense:

$$ f \cdot g = \sum_{j=0}^{\infty} \left(\sum_{r=0}^{j} \binom{j/n}{r/n} c_r b_{j-r} \right) w_j^{(n)}, \tag{4} $$

where $\binom{\lambda}{\mu} = \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\lambda-\mu+1)}$ denotes the generalized binomial coefficient.

The set ${}^{C}\mathbb{F}_n$, together with these operations, forms a commutative algebra over the field of complex numbers, denoted as ${}^{C}\mathcal{F}_n$ [33].

Definition 2.4. The generating function $\Phi(t)$ associated with the fractional power series $y(x) \in {}^{C}\mathcal{F}_n$ is defined by the transformation:

$$ \Phi(t) = \sum_{j=0}^{\infty} \gamma_j t^j, \quad \text{where } \gamma_j = \frac{c_j}{\Gamma\left(\frac{j}{n}+1\right)}. \tag{5} $$

By substitution, it is straightforward to verify that the relationship between the fractional series in the original x-domain and the corresponding generating function in the transformed t-domain is given by:

$$y(x) = \Phi(x^{1/n}). \tag{6}$$

It is important to note that the mapping between the fractional power series $y(x) \in {}^{C}\mathcal{F}_n$ and its generating function $\Phi(t)$ constitutes an algebra isomorphism [33]. Consequently, the multiplication of two fractional power series corresponds to the standard multiplication of their respective generating functions:

$$z(x) = y_1(x) \cdot y_2(x) \iff \Phi_z(t) = \Phi_1(t) \cdot \Phi_2(t). \tag{7}$$

This isomorphism naturally extends to any analytic function F. Therefore, for a nonlinear term $F(y(x))$ present in a fractional differential equation, the equivalent representation in the generating function domain simplifies to $F(\Phi(t))$.

2.2. Caputo fractional derivative operator

Definition 2.5. The Caputo fractional differentiation operator ${}^{C}\mathbf{D}^{(1/n)}$ is defined by its action on the elements of the fractional basis as follows:

$${}^{C}\mathbf{D}^{(1/n)} w_j^{(n)} = \begin{cases} w_{j-1}^{(n)}, & j \geq 1; \\ 0, & j = 0. \end{cases} \tag{8}$$

This operator can be generalized to higher orders by applying it iteratively.

Definition 2.6. Applying the operator k times yields the higher-order operator $\left({}^{C}\mathbf{D}^{(1/n)}\right)^k$, which shifts the basis indices by k:

$$\left({}^{C}\mathbf{D}^{(1/n)}\right)^k w_j^{(n)} = \begin{cases} w_{j-k}^{(n)}, & j \geq k; \\ 0, & j < k. \end{cases} \tag{9}$$

Consequently, the application of the operator $\left({}^{C}\mathbf{D}^{(1/n)}\right)^k$ to an arbitrary fractional power series $y(x) = \sum_{j=0}^{\infty} c_j w_j^{(n)}$ within the defined algebra results in the following expression:

$$\left({}^{C}\mathbf{D}^{(1/n)}\right)^k y(x) = \sum_{j=k}^{\infty} c_j w_{j-k}^{(n)} = \sum_{j=0}^{\infty} c_{j+k} w_j^{(n)}. \tag{10}$$

It is important to emphasize that when operating within the fractional power series algebra ${}^{C}\mathcal{F}_n$, the derivative operation defined by (9) is strictly equivalent to the classical integral definition of the Caputo derivative ${}^{C}\mathbf{D}^{(\alpha)}$ given in (1), provided that the derivative order is rational, specifically $\alpha = \frac{k}{n}$ [33]. It should be noted that the classical integral definition remains more general, as it natively supports irrational derivative orders. Because the framework utilized in this study relies on rational fractional bases, irrational orders remain outside the scope of the proposed techniques.

It is also worth noting that the nth iteration of the Caputo fractional differentiation operator, $\left({}^{C}\mathbf{D}^{(1/n)}\right)^n$, is not identical to the standard first-order derivative $\frac{\mathrm{d}}{\mathrm{d}x}$. As demonstrated in [33], this fundamental distinction arises because the fractional operator $\left({}^{C}\mathbf{D}^{(1/n)}\right)^n$ acts specifically on Caputo fractional power series composed of the fractional basis elements $w_j^{(n)}$. In contrast, the classical integer-order derivative operates strictly on standard Taylor series containing only integer powers of x.

3. Reduction of nonlinear Caputo fractional systems to ordinary differential equations

While the fractional power series algebra has been successfully utilized to systematically reduce scalar fractional equations to equivalent first-order ODEs [33,34], the following derivations establish the novel extension of this mathematical framework to nonlinear systems of Caputo FDEs.

Consider a system of m coupled nonlinear Caputo FDEs:

$$\left({}^{C}\mathbf{D}^{(1/n)}\right)^n y_i(x) = F_i(y_1, \ldots, y_m), \quad i = 1, \ldots, m, \tag{11}$$

subject to the initial conditions for $k = 0, \ldots, n-1$:

$$\left({}^{C}\mathbf{D}^{(1/n)}\right)^k y_i(0) = s_{i,k}, \tag{12}$$

where $y_1, \ldots, y_m \in {}^{C}\mathcal{F}_n$ represent the unknown functions.

It is important to note, that while the standard Caputo derivative ${}^{C}\mathbf{D}^{(\alpha)}$ typically requires only classical, integer-order initial conditions, the system defined in (11) utilizes the sequential fractional operator $\left({}^{C}\mathbf{D}^{(1/n)}\right)^n$, for which the intermediate fractional states at the origin, denoted by $s_{i,k} = \left({}^{C}\mathbf{D}^{(1/n)}\right)^k y_i(0)$, must be specified. In the context of the physical models analyzed in this study, these fractional initial conditions define the initial memory state of the system.

In this section, a mapping from the fractional domain to an integer-order domain is constructed, effectively reducing the fractional system to a system of ordinary differential equations.

The reduction process begins by expanding the unknown functions $y_i(x)$ within the fractional basis as $y_i(x) = \sum_{j=0}^{\infty} c_{i,j} w_j^{(n)}$. Each function y_i is associated with a generating function $\Phi_i(t) = \sum_{j=0}^{\infty} \gamma_{i,j} t^j$, where $\gamma_{i,j} = \frac{c_{i,j}}{\Gamma(j/n+1)}$.

Applying the operator property from Eq. (10) to the derivative term on the left-hand side of Eq. (11) yields:

$$\left({}^{C}\mathbf{D}^{(1/n)}\right)^n y_i(x) = \sum_{k=0}^{\infty} c_{i,k+n} w_k^{(n)}. \tag{13}$$

The coefficients $c_{i,k+n}$ can be expressed in terms of the generating function coefficients γ:

$$c_{i,k+n} = \gamma_{i,k+n} \Gamma\left(\frac{k+n}{n}+1\right) = \gamma_{i,k+n}\left(\frac{k}{n}+1\right)\Gamma\left(\frac{k}{n}+1\right). \tag{14}$$

Consequently, the coefficient of the expansion (13), corresponding to the basis vector $w_k^{(n)}$ for the fractional derivative term reads:

$$\frac{k+n}{n}\gamma_{i,k+n}\Gamma\left(\frac{k}{n}+1\right). \tag{15}$$

Next, the expansion coefficients of the (11) right-hand side term $F_i(y_1, \ldots, y_m)$ must be determined in the basis $w_k^{(n)}$. The image of this term in the generating function domain is considered. Because the mapping between the original series $y(x)$ and the generating function $\Phi(t)$ constitutes an algebra isomorphism, the generating function corresponding to the composite nonlinear term simplifies to F_i evaluated at the respective generating functions $\Phi_1, \ldots, \Phi_m$:

$$G_i(t) = F_i(\Phi_1(t), \ldots, \Phi_m(t)). \tag{16}$$

It is assumed that the nonlinear vector function F_i is analytic in its arguments. This analyticity requirement is a fundamental prerequisite for this algebraic reduction methodology, as it ensures the existence of a rigorously convergent multi-variable power series expansion [2]. Note that for the specific physical models analyzed further in this study (such as the multiplicatively coupled fractional Riccati system), the nonlinearities are polynomial, which are analytic everywhere in the complex plane. Consequently, this necessary analyticity condition is globally and naturally satisfied without the need for local approximation. This property permits the expansion of $G_i(t)$ as a standard power series in the transformed variable t:

$$G_i(t) = \sum_{k=0}^{\infty} \beta_{i,k} t^k. \tag{17}$$

This series is then mapped back to the fractional domain ${}^{C}\mathcal{F}_n$ by utilizing the inverse of the generating function definition:

$$F_i(y_1, \ldots, y_m) = \sum_{k=0}^{\infty} \beta_{i,k}\Gamma\left(\frac{k}{n}+1\right) w_k^{(n)}. \tag{18}$$

Therefore, the expansion coefficient associated with the basis vector $w_k^{(n)}$ for the nonlinear term on the right-hand side of (11) is given by:

$$\beta_{i,k}\Gamma\left(\frac{k}{n}+1\right). \tag{19}$$

By equating the corresponding coefficients of the basis functions $w_k^{(n)}$ derived from the derivative and nonlinear components (15) and (19), the following algebraic identity is obtained:

$$\frac{k+n}{n}\gamma_{i,k+n}\Gamma\left(\frac{k}{n}+1\right) = \beta_{i,k}\Gamma\left(\frac{k}{n}+1\right), \tag{20}$$

which simplifies into a recurrence relation for the generating function coefficients:

$$(k+n)\gamma_{i,k+n} = n\beta_{i,k}. \tag{21}$$

To transform this algebraic recurrence relation back into a continuous differential equation, the relation is multiplied by t^k and summed over all non-negative integers k:

$$\sum_{k=0}^{\infty}(k+n)\gamma_{i,k+n}t^k = n\sum_{k=0}^{\infty}\beta_{i,k}t^k. \tag{22}$$

Given (17), the right-hand side of (22) evaluates to:

$$n\sum_{k=0}^{\infty}\beta_{i,k}t^k = nG_i(t) = nF_i(\Phi_1, \ldots, \Phi_m). \tag{23}$$

For the left-hand side of (22), a change of index $j = k + n$ as well as factoring out the term $t^{-(n-1)}$ yields:

$$\sum_{k=0}^{\infty}(k+n)\gamma_{i,k+n}t^k = \frac{1}{t^{n-1}}\sum_{j=n}^{\infty} j\gamma_{i,j}t^{j-1}. \tag{24}$$

Note that the first-order derivative of the generating function $\Phi_i(t) = \sum_{j=0}^{\infty}\gamma_{i,j}t^j$ is given by:

$$\frac{\mathrm{d}\Phi_i}{\mathrm{d}t} = \sum_{j=1}^{\infty} j\gamma_{i,j}t^{j-1}. \tag{25}$$

The summation index of the left-hand side of (24) starts at $j = n$, omitting the first $n-1$ terms of the full derivative expansion. The sum is therefore rewritten as the complete derivative minus the truncated initial terms:

$$\sum_{j=n}^{\infty} j\gamma_{i,j}t^{j-1} = \sum_{j=1}^{\infty} j\gamma_{i,j}t^{j-1} - \sum_{j=1}^{n-1} j\gamma_{i,j}t^{j-1}, \tag{26}$$

where the subtracted term $P_i(t) = \sum_{j=1}^{n-1} j\gamma_{i,j}t^{j-1}$ represents a polynomial entirely determined by the initial conditions of the original fractional system (11). Thus, (24) can be simplified to

$$\frac{1}{t^{n-1}}\left(\frac{\mathrm{d}\boldsymbol{\Phi}_i}{\mathrm{d}t} - P_i(t)\right). \tag{27}$$

Equating the obtained expressions for the derivative and nonlinear components (23) and (24) results in the following ODE system:

$$\frac{1}{t^{n-1}}\left(\frac{\mathrm{d}\boldsymbol{\Phi}_i}{\mathrm{d}t} - P_i(t)\right) = nF_i(\boldsymbol{\Phi}_1, \dots, \boldsymbol{\Phi}_m), \tag{28}$$

which can be rearranged to the following form:

$$\frac{\mathrm{d}\boldsymbol{\Phi}_i}{\mathrm{d}t} = nt^{n-1}F_i(\boldsymbol{\Phi}_1, \dots, \boldsymbol{\Phi}_m) + P_i(t), \quad i = 1, \dots, m. \tag{29}$$

Here, $P_i(t)$ represents the memory effects of the fractional system defined by the initial conditions:

$$P_i(t) = \sum_{k=1}^{n-1} k\gamma_{i,k}t^{k-1} = \sum_{k=1}^{n-1} k\frac{s_{i,k}}{\Gamma(k/n+1)}t^{k-1}. \tag{30}$$

Once the reduced integer-order system is solved for $\boldsymbol{\Phi}_i(t)$, the solutions to the original fractional differential equations are recovered by applying the substitution $t = x^{1/n}$, which yields $y_i(x) = \boldsymbol{\Phi}_i(x^{1/n})$.

Theorem 3.1. *Let $y_1, \dots, y_m \in {}^{C}\mathcal{F}_n$ be analytical solutions to the system of Caputo fractional differential equations*

$$\left({}^{C}\mathbf{D}^{(1/n)}\right)^n y_i(x) = F_i(y_1, \dots, y_m), \quad i = 1, \dots, m, \tag{31}$$

subject to the initial conditions $\left({}^{C}\mathbf{D}^{(1/n)}\right)^k y_i(0) = s_{i,k}$ for $k = 0, \dots, n-1$. This fractional system is equivalent to the following system of first-order ODEs:

$$\frac{\mathrm{d}\boldsymbol{\Phi}_i}{\mathrm{d}t} = nt^{n-1}F_i(\boldsymbol{\Phi}_1, \dots, \boldsymbol{\Phi}_m) + \sum_{k=1}^{n-1} k\frac{s_{i,k}}{\Gamma(k/n+1)}t^{k-1}, \quad i = 1, \dots, m, \tag{32}$$

subject to the initial conditions $\boldsymbol{\Phi}_i(0) = s_{i,0}$. The solutions to the original fractional system are recovered via the time transformation $y_i(x) = \boldsymbol{\Phi}_i(x^{1/n})$.

Proof. The proof follows directly from the derivations presented in this section. A schematic diagram of the presented reduction methodology is illustrated in Fig. 1. □

Remark 3.1. Note that the mechanism by which the inherently non-local memory of the Caputo operator (a global convolution integral) is transformed into a local non-autonomous forcing representation in the integer-order domain is a direct consequence of the generating function isomorphism utilized in Theorem 3.1. By expressing the system dynamics within the fractional power series ring ${}^{C}\mathcal{F}_n$, the fractional convolution integral is evaluated analytically term-by-term. Through the change of variables $t = x^{1/n}$, the distributed historical memory is not truncated or approximated, rather, it is deterministically re-encoded. Specifically, the infinite memory tail is algebraically compressed into the explicit polynomial time-dependence of the resulting ordinary differential equations, and scaled by the system's fractional initial conditions $s_{i,k}$. Thus, the global integro-differential nature of the fractional medium is exactly mapped to a local, deterministic forcing term within the transformed temporal coordinate space.

It is important to distinguish the algebraic reduction methodology presented here from the heuristic fractional complex transform occasionally found in the literature [43]. As correctly noted in recent critical studies [44], such heuristic transforms often fail because they implicitly assume the validity of the chain rule for fractional derivatives, which generally does not hold. The proposed methodology, however, does not rely on the fractional chain rule. Instead, the reduction is exact and rigorously justified by the algebra isomorphism established between the fractional power series domain ${}^{C}\mathcal{F}_n$ and the standard generating function domain (as detailed in Section 2).

Furthermore, the applied coordinate transformation $t = x^{1/n}$ acts as an unfolding map that resolves the algebraic branching singularity of the fractional solutions at the origin $x = 0$. It maps the multi-sheeted Riemann surface of the fractional domain onto a regular, flat complex plane t, where standard integer-order differentiation and analytic continuation are strictly valid. It should be noted that while this exact isomorphism is valid for the operator $\left({}^{C}\mathbf{D}^{(1/n)}\right)^n$, for a general operator $\left({}^{C}\mathbf{D}^{(1/n)}\right)^k$ with $k \neq n$, the reduction would yield an infinite series tail, rendering the method approximate unless a closed-form representation for the residual tail can be identified.

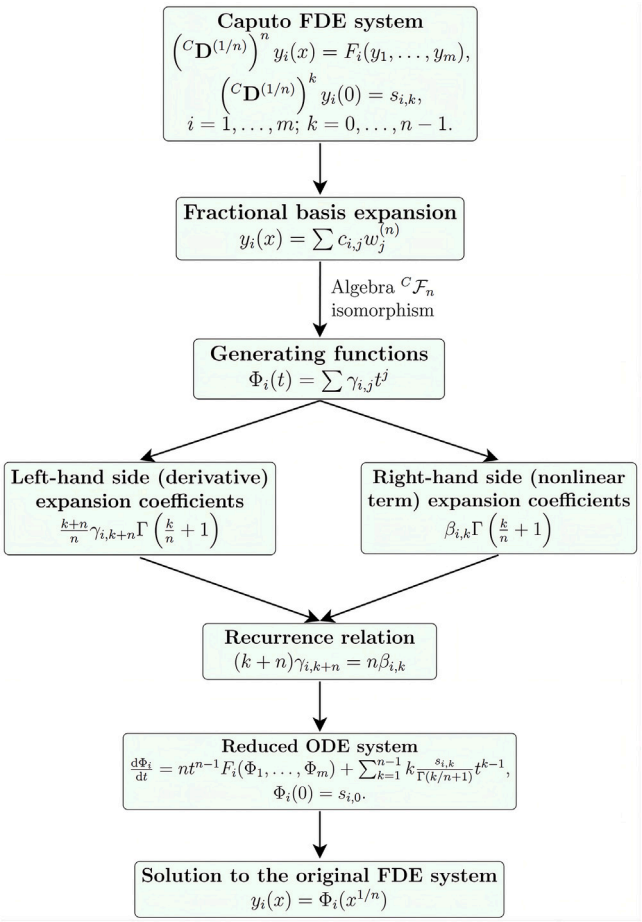

Fig. 1. Schematic diagram illustrating the proposed methodology for solving a Caputo FDE system.

It is also worth contextualizing the practical utility of the isomorphism established in Theorem 3.1. Within this framework, the fractional formulation (31) represents the primary problem of interest, governing the actual physical dynamics and memory effects under investigation. However, the inherent non-local nature of fractional operators often makes the direct resolution of such FDEs challenging, typically necessitating the development of specialized, FDE-specific techniques. To circumvent these difficulties, this methodology introduces the reduction to the equivalent integer-order system (32) as an intermediate analytical tool. By reducing the FDE system to an ODE system, this methodology bypasses the need for novel fractional solvers and allows for the application of the extensive, well-established mathematical techniques already available for ODEs. Once the equivalent ODE system is solved, the reverse transformation maps the solution back to the fractional domain. Ultimately, this detour through the integer-order space facilitates the discovery of novel exact solutions and dynamic behaviors within the original FDE system that could otherwise be mathematically intractable to derive directly.

3.1. Reduction under zero fractional initial conditions

The behavior of the reduced system (29) under the assumption that all fractional-order initial conditions are equal to zero is examined in this section. This scenario corresponds to a physical state where the memory effects induced by the fractional initial states are entirely absent.

Let:

$$s_{i,0} = y_i(0) \in \mathbb{R}, \quad \text{and} \quad s_{i,k} = 0 \quad \text{for } k = 1, \ldots, n-1. \tag{33}$$

Under the assumption of vanishing fractional initial values, the coefficients of the polynomial $P_i(t)$ (29) become identically zero:

$$\gamma_{i,k} = 0, \quad \text{for } k = 1, \ldots, n-1, \tag{34}$$

leading to $P_i(t) = 0$.

Substituting $P_i(t) = 0$ into the reduced system (29) derived previously simplifies the equations to:

$$\frac{\mathrm{d}\boldsymbol{\Phi}_i}{\mathrm{d}t} = nt^{n-1} F_i(\boldsymbol{\Phi}_1, \dots, \boldsymbol{\Phi}_m). \tag{35}$$

It can be shown that this simplified system is equivalent to the system obtained by replacing the fractional derivative operator $\left({}^{C}\mathbf{D}^{(1/n)}\right)^n$ in Eq. (11) with the standard first-order derivative $\frac{\mathrm{d}}{\mathrm{d}x}$:

$$\frac{\mathrm{d}y_i}{\mathrm{d}x} = F_i(y_1, \dots, y_m). \tag{36}$$

Using the previously established isomorphism $y_i(x) = \boldsymbol{\Phi}_i(x^{1/n})$, the change of variable $x = t^n$ is applied to the integer-order system (36). By the chain rule, the derivative transforms as follows:

$$\frac{\mathrm{d}y_i}{\mathrm{d}x} = \frac{\mathrm{d}y_i}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{\mathrm{d}\boldsymbol{\Phi}_i}{\mathrm{d}t}\frac{1}{nt^{n-1}}. \tag{37}$$

Substituting this derivative back into Eq. (36) yields:

$$\frac{1}{nt^{n-1}}\frac{\mathrm{d}\boldsymbol{\Phi}_i}{\mathrm{d}t} = F_i(\boldsymbol{\Phi}_1, \dots, \boldsymbol{\Phi}_m), \tag{38}$$

which can be rearranged to the simplified reduced system presented in Eq. (35).

Thus, if all fractional initial conditions are identically zero, solving the fractional system (11) is equivalent to solving the standard integer-order ordinary differential equation system (36) under the coordinate transformation $x = t^n$.

It is important to emphasize that the equivalence established here between the repeated fractional operator $\left({}^{C}\mathbf{D}^{(1/n)}\right)^n$ and the classical first-order derivative $\frac{\mathrm{d}}{\mathrm{d}x}$ does not contradict the fundamental differences outlined in Section 2.2. Rather, this equivalence emerges as a consequence of the vanishing fractional initial conditions, which completely eliminate the memory polynomial $P_i(t)$. Within this mathematical framework, the fractional initial conditions act as the primary drivers of memory. If even a single fractional initial condition were non-zero, the memory polynomial would persist, and this direct equivalence to a local first-order derivative would not apply. Therefore, while the fractional operator $\left({}^{C}\mathbf{D}^{(1/n)}\right)^n$ is fundamentally broader for a general system of FDEs, it coincides with the classical first-order derivative $\frac{\mathrm{d}}{\mathrm{d}x}$ under the specific scenario where the memory effects induced by the fractional initial conditions are absent.

4. Impact of fractional memory on multiplicatively coupled Riccati systems

The framework established in the preceding sections provides a mechanism for mapping memory effects inherent to fractional-order systems into a time-dependent polynomial forcing term. In this section, this reduction technique is applied to investigate the impact of fractional memory on multiplicatively coupled Riccati systems.

Physically, the multiplicatively coupled Riccati system serves as a fundamental macroscopic model for describing interacting nonlinear modes. In standard integer-order dynamics, Riccati equations frequently emerge from the reduction of nonlinear partial differential equations (such as the Korteweg–de Vries or nonlinear Schrödinger equations) that govern wave propagation, plasma physics, and fluid dynamics. The quadratic nonlinearities natively capture essential physical mechanisms: the y_i^2 terms represent self-interaction or wave steepening, while the $y_1 y_2$ terms model the cross-coupling or energy transfer between distinct wave modes. By transitioning to a fractional-order framework, this mathematical model is explicitly extended to describe anomalous, non-Markovian transport in complex media, such as viscoelastic fluids or disordered dielectrics. In these environments, the Caputo fractional derivative represents a fading memory effect, where the medium resists instantaneous changes, meaning the current propagation of the wave is continuously modified by the residual memory of its historical trajectory.

The main goal of this analysis is to explore the capacity of fractional memory effects to deform existing stable structures (such as solitary waves) and their potential to generate new, exactly solvable analytical states under specific conditions.

Consider the following system of multiplicatively coupled Caputo fractional Riccati equations of order $n = 2$ with respect to functions $y_1, y_2 \in {}^{C}\mathcal{F}_2$:

$$\begin{cases} \left({}^{C}\mathbf{D}^{(1/2)}\right)^2 y_1(x) = a_0 + a_1 y_1 + a_2 y_1^2 + a_3 y_1 y_2, \\ \left({}^{C}\mathbf{D}^{(1/2)}\right)^2 y_2(x) = b_0 + b_1 y_2 + b_2 y_2^2 + b_3 y_1 y_2, \end{cases} \tag{39}$$

subject to the initial conditions $y_i(0) = s_{i,0}$ and ${}^{C}\mathbf{D}^{(1/2)} y_i(0) = s_{i,1}$ for $i = 1, 2$.

Applying the general reduction procedure derived in Section 3, the fractional system is mapped to the generating function domain. The history polynomials $P_i(t)$ for $n = 2$ simplify to constant terms $\gamma_{i,1} = 2s_{i,1}/\sqrt{\pi}$. Consequently, the original fractional system is equivalent to the following system of first-order ODEs:

$$\begin{cases} \frac{\mathrm{d}\boldsymbol{\Phi}_1}{\mathrm{d}t} = 2t\left(a_0 + a_1\boldsymbol{\Phi}_1 + a_2\boldsymbol{\Phi}_1^2 + a_3\boldsymbol{\Phi}_1\boldsymbol{\Phi}_2\right) + \gamma_{1,1}, \\ \frac{\mathrm{d}\boldsymbol{\Phi}_2}{\mathrm{d}t} = 2t\left(b_0 + b_1\boldsymbol{\Phi}_2 + b_2\boldsymbol{\Phi}_2^2 + b_3\boldsymbol{\Phi}_1\boldsymbol{\Phi}_2\right) + \gamma_{2,1}, \end{cases} \tag{40}$$

subject to the initial conditions $\boldsymbol{\Phi}_i(0) = s_{i,0}$. The solutions to the original fractional system are obtained via the transformation $y_i(x) = \boldsymbol{\Phi}_i(\sqrt{x})$.

4.1. Deformation of solitary solutions under memory effects

To illustrate the effect of fractional initial conditions on the system dynamics, a baseline configuration is first established. In the absence of fractional memory, meaning the history terms $\gamma_{1,1}$ and $\gamma_{2,1}$ are strictly zero, the reduced system (40) is equivalent to the standard integer-order system under the time transformation $x = t^2$:

$$\begin{cases} \frac{\mathrm{d}y_1}{x} = a_0 + a_1 y_1 + a_2 y_1^2 + a_3 y_1 y_2, \\ \frac{\mathrm{d}y_2}{x} = b_0 + b_1 y_2 + b_2 y_2^2 + b_3 y_1 y_2. \end{cases} \tag{41}$$

It is known that such an unperturbed integer-order system admits bright and dark solitary wave solutions under specific parameter constraints [35]. Thus, to investigate how fractional memory deforms these solitary wave structures, the following parameters were selected:

$$a_0 = \frac{136}{11}, \quad a_1 = -\frac{828}{319}, \quad a_2 = \frac{29}{187}, \quad a_3 = -\frac{550}{1479},$$
$$b_0 = -\frac{51}{29}, \quad b_1 = \frac{345}{319}, \quad b_2 = -\frac{550}{1479}, \quad b_3 = \frac{29}{187}.$$

The integer-order initial conditions are fixed at $y_1(0) = 12$ and $y_2(0) = 3$.

When fractional memory is introduced such that $\gamma_{i,1} \neq 0$, the history polynomial acts as a non-autonomous forcing term which breaks the exact integrability of the solitary wave regime and deforms the solution trajectories. A computational experiment is performed to investigate the influence of these fractional initial conditions on the solitary wave structures.

To visualize the deformation of the solitary solutions under varying memory conditions, a two-dimensional parameter space for the fractional initial conditions $(\gamma_{1,1}, \gamma_{2,1})$ is explored. The parameters are sampled from three concentric circles centered at the origin, with radii $R = 1/3$, $R = 2/3$, and $R = 1$. On each circle, eight equidistant points are selected, representing distinct directional combinations of memory perturbations. The origin $(0, 0)$ corresponds to the unperturbed integer-order system where fractional memory is entirely absent. The resulting trajectories are obtained by numerically integrating the reduced system of ODEs (40) and applying the time transformation $x = t^2$.

The dynamics generated by this parameter sampling are illustrated in Fig. 2. Panel (a) displays the selected parameter configurations within the $(\gamma_{1,1}, \gamma_{2,1})$ space. The subsequent panels illustrate the dynamics of the first component $y_1(x)$ and the second component $y_2(x)$ for the perturbation radii $R = 1/3$ (Panel b), $R = 2/3$ (Panel c), and $R = 1$ (Panel d). In all trajectory subplots, the thick black line represents the unperturbed solitary wave, while the thinner colored lines correspond to the deformed solutions governed by the parameters on the respective circles.

An analysis of the generated trajectories in Fig. 2 shows that the introduction of fractional memory smoothly deforms the solitary wave structures, creating a continuous envelope of perturbed states around the standard localized behavior. The unperturbed integer-order system exhibits a distinct local minimum for the first component before gradually ascending, while the second component rises to a pronounced peak amplitude. When non-zero fractional initial conditions are introduced, the resulting non-autonomous forcing fundamentally alters both the amplitude and the width of these localized waves. As the magnitude of the memory perturbation increases – moving from $R = 1/3$ in Panel (b) to $R = 1$ in Panel (d) – the deviation from the baseline trajectory expands, significantly stretching or compressing the wave profiles vertically depending on the angular direction of the perturbation. Furthermore, the memory terms alter the asymptotic decay rates of the trajectories, widening the pulse structures and shifting their recovery paths. This demonstrates that fractional memory acts as a continuous shape modulator, breaking the exact integrability and precise symmetry of the original integer-order solitons while preserving their fundamentally bounded and localized nature.

These visual patterns can be linked to the underlying structure of the reduced ordinary differential equation system (40). When the reduced system is mapped back to the physical domain via the transformation $t = \sqrt{x}$, the constant memory parameters $\gamma_{i,1}$ are reflected in the derivative with respect to x as non-autonomous forcing terms of the form $\gamma_{i,1}/(2\sqrt{x})$. Consequently, the fractional memory induces an infinite-gradient singularity strictly at the initial time instant as x approaches zero. The memory does not simply shift the curves, it delivers an immediate directional perturbation at the origin that rapidly decays as the spatial coordinate increases, allowing the standard Riccati nonlinearities to dictate the later asymptotic relaxation.

Furthermore, an asymmetric sensitivity to the fractional memory between the two system components can be observed across all perturbation radii in Fig. 2. The visual deviations from the unperturbed state are considerably more pronounced in the second component than in the first. This discrepancy arises directly from the relative scale of the state variables. Since the initial value of the first component is an order of magnitude larger than that of the second, the internal quadratic self-interaction term $a_2 y_1^2$ dominates the local dynamics of the first equation. This large nonlinear inertia acts as a localized stiffness, making the first component highly resistant to the additive memory perturbation. In contrast, the smaller absolute scale of the second component renders its governing dynamics more sensitive to both the direct memory perturbation and the multiplicative cross-coupling feedback from the first component.

It is important to distinguish this memory-induced deformation from the closely related concept of fractal solitary waves. Fractal solitary waves typically describe localized structures propagating through geometrically complex, non-Euclidean media – such as porous materials, fractal lattices, or rough physical boundaries – where the deformation is strictly driven by the non-integer spatial dimension of the physical environment itself. Conversely, the memory-induced deformation investigated in this study occurs within a standard, smooth Euclidean domain. Here, the structural warping of the solitary wave is not caused by spatial boundary constraints,

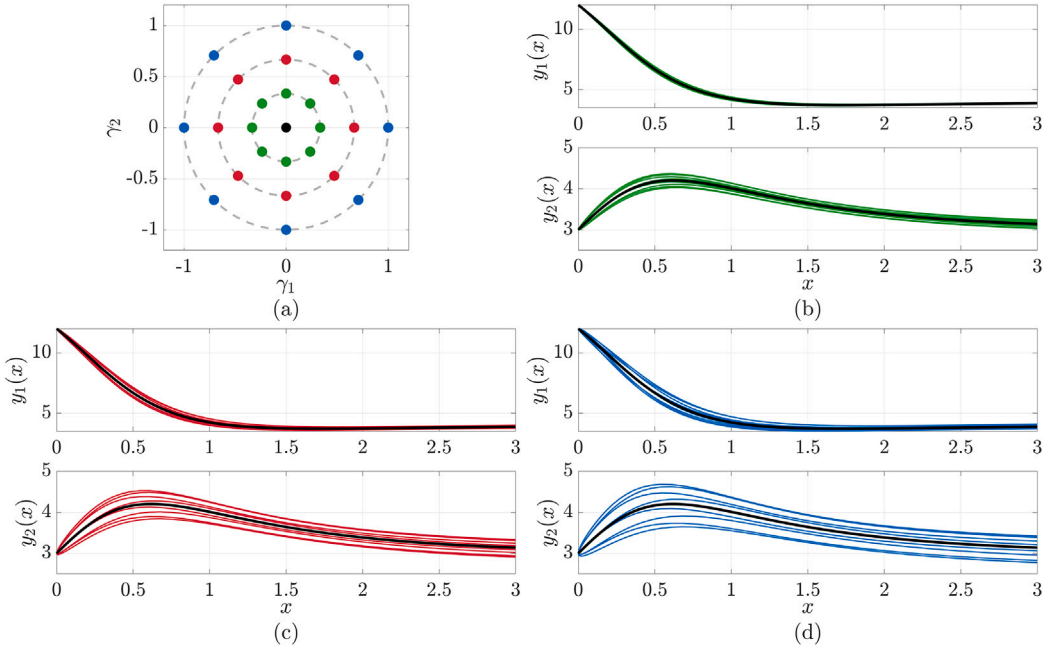

Fig. 2. Deformation of the coupled Riccati system solitary solutions under varying memory conditions. Panel (a) illustrates the parameter space of the fractional initial conditions $(\gamma_{1,1}, \gamma_{2,1})$, featuring 24 points distributed across three concentric circles with radii $R = 1/3$, $R = 2/3$, and $R = 1$, along with the origin $(0, 0)$. Panels (b), (c), and (d) display the corresponding trajectories for $y_1(x)$ (top subplots) and $y_2(x)$ (bottom subplots) for the radii $R = 1/3$ (green lines), $R = 2/3$ (red lines), and $R = 1$ (blue lines), respectively. The thick black line in all panels denotes the unperturbed integer-order solitary solution. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

but rather by the non-local, integro-differential nature of the fractional operator, which acts as an intrinsic, history-dependent forcing term that continuously modulates the wave dynamics.

Physically, the visual deformations observed in Fig. 2, specifically the distinct alterations in wave amplitude, width, and the loss of symmetric asymptotic decay, directly represent the continuous drag exerted by the medium's fading memory. As the wave propagates through the complex fractional medium, it is constantly forced by its own non-local history (encoded via the fractional initial conditions). Therefore, the geometric warping of the solitary profile seen in the graphs is not merely a mathematical artifact, but the physical effect of the medium's memory modulating the wave's localized energy distribution over time.

4.2. Exact analytical solutions of the reduced system (40)

In the preceding subsection, it was demonstrated that introducing fractional memory to the parameter configuration supporting solitary waves acts as a non-autonomous perturbation, deforming the solutions and compromising their exact integrability. However, the loss of integrability in this specific case does not preclude the existence of exact solutions elsewhere. Analytical solutions can still be constructed in the presence of memory when certain constraints are satisfied, specifically within a parameter subspace where memory effects are compensated by the nonlinear terms. This subsection details the derivation of such analytical solutions for the reduced ODE system (40).

To solve the reduced system (40) analytically, a new independent variable is first introduced as $\tau = t^2$ for $t > 0$. Using the chain rule, the differential operator transforms as

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\tau} = 2t\frac{\mathrm{d}}{\mathrm{d}\tau} = 2\sqrt{\tau}\frac{\mathrm{d}}{\mathrm{d}\tau}.$$

Substituting this operator into (40) and dividing both equations by $2\sqrt{\tau}$ yields the transformed system:

$$\begin{cases} \frac{\mathrm{d}\Phi_1}{\mathrm{d}\tau} = a_0 + a_1\Phi_1 + a_2\Phi_1^2 + a_3\Phi_1\Phi_2 + \frac{\gamma_{1,1}}{2\sqrt{\tau}}, \\ \frac{\mathrm{d}\Phi_2}{\mathrm{d}\tau} = b_0 + b_1\Phi_2 + b_2\Phi_2^2 + b_3\Phi_1\Phi_2 + \frac{\gamma_{2,1}}{2\sqrt{\tau}}. \end{cases} \tag{42}$$

Next, the obtained system (42) is linearized via substitution. It is assumed that the variables Φ_1 and Φ_2 are quotients of functions originating from a higher-dimensional linear space defined by (u, v, w), such that

$$\Phi_1(\tau) = \frac{u(\tau)}{w(\tau)}, \quad \Phi_2(\tau) = \frac{v(\tau)}{w(\tau)}. \tag{43}$$

Differentiating Φ_1 with respect to τ yields:

$$\frac{\mathrm{d}\Phi_1}{\mathrm{d}\tau} = \frac{u'w - uw'}{w^2} = \frac{u'}{w} - \Phi_1 \frac{w'}{w}. \tag{44}$$

To map the nonlinear terms of (42) into this linear framework, the auxiliary denominator $w(\tau)$ is defined as

$$w'(\tau) = -a_1 w(\tau) - a_2 u(\tau) - a_3 v(\tau). \tag{45}$$

Substituting (45) into (44) yields

$$\begin{aligned}\frac{\mathrm{d}\Phi_1}{\mathrm{d}\tau} &= \frac{u'}{w} - \Phi_1 \cdot \left(\frac{-a_1 w - a_2 u - a_3 v}{w}\right) \\ &= \frac{u'}{w} + a_1\Phi_1 + a_2\Phi_1^2 + a_3\Phi_1\Phi_2.\end{aligned} \tag{46}$$

Equating this result with the right-hand side of the first equation in (42) cancels the terms containing Φ_1 and Φ_2, isolating the required linear differential equation for $u(\tau)$:

$$u'(\tau) = \left(a_0 + \frac{\gamma_{1,1}}{2\sqrt{\tau}}\right) w(\tau). \tag{47}$$

An identical procedure is applied to the second variable. Differentiating Φ_2 and substituting the defined expression for $w'(\tau)$ yields:

$$\begin{aligned}\frac{\mathrm{d}\Phi_2}{\mathrm{d}\tau} &= \frac{v'}{w} - \Phi_2 \cdot \left(\frac{-a_1 w - a_2 u - a_3 v}{w}\right) \\ &= \frac{v'}{w} + a_1\Phi_2 + a_2\Phi_1\Phi_2 + a_3\Phi_2^2.\end{aligned} \tag{48}$$

Comparing the derived expression (48) with the second equation of (42) reveals that the substitution is valid if and only if the following constraints on the system parameters hold:

$$a_1 = b_1,\ a_2 = b_3,\ a_3 = b_2 \tag{49}$$

If the above conditions are satisfied, the remaining terms yield the linear relation for $v(\tau)$:

$$v'(\tau) = \left(b_0 + \frac{\gamma_{2,1}}{2\sqrt{\tau}}\right) w(\tau). \tag{50}$$

Thus, under the constraints (49), the nonlinear system (42) maps exactly to the following linear system:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 & 0 & a_0 + \frac{\gamma_{1,1}}{2\sqrt{\tau}} \\ 0 & 0 & b_0 + \frac{\gamma_{2,1}}{2\sqrt{\tau}} \\ -a_2 & -a_3 & -a_1 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix}. \tag{51}$$

To solve the system (51), the variable $w(\tau)$ is decoupled by differentiating the third equation $w' = -a_1 w - a_2 u - a_3 v$ a second time:

$$w'' = -a_1 w' - a_2 u' - a_3 v'.$$

Substituting u' and v' from the system (51) and rearranging the terms produces a linear second-order ordinary differential equation:

$$w''(\tau) + a_1 w'(\tau) + \left(C_0 + C_1\tau^{-1/2}\right) w(\tau) = 0, \tag{52}$$

where $C_0 = a_2 a_0 + a_3 b_0$ and $C_1 = \frac{1}{2}(a_2\gamma_{1,1} + a_3\gamma_{2,1})$.

To solve Eq. (52) analytically, the first derivative term $a_1 w'(\tau)$ is eliminated by introducing the transformation

$$w(\tau) = e^{-\frac{a_1}{2}\tau} f(\tau) \tag{53}$$

Substituting (53) into (52) results in:

$$e^{-\frac{a_1}{2}\tau}\left[\left(f'' - a_1 f' + \frac{a_1^2}{4} f\right) + a_1\left(f' - \frac{a_1}{2} f\right) + \left(C_0 + C_1\tau^{-1/2}\right) f\right] = 0. \tag{54}$$

Divided by the exponential as well as grouping terms by derivative order yields an equation that structurally resembles a stationary Schrödinger equation with a combined constant and inverse square-root potential:

$$f''(\tau) + \left[\tilde{C}_0 + C_1\tau^{-1/2}\right] f(\tau) = 0, \tag{55}$$

where $\tilde{C}_0 = C_0 - a_1^2/4$ represents the effective energy. In the general case where $\tilde{C}_0 \neq 0$, this equation corresponds to the biconfluent Heun equation, the solutions of which are expressed in terms of biconfluent Heun functions. These functions possess a significantly more involved mathematical structure compared to standard hypergeometric or Whittaker functions. To avoid this

structural complexity and derive a solution in terms of standard Bessel functions, the additional solvability constraint $\tilde{C}_0 = 0$ is imposed, which yields:

$$a_2 a_0 + a_3 b_0 = \frac{a_1^2}{4}. \tag{56}$$

When (56) is satisfied, Eq. (55) simplifies to:

$$f''(\tau) + C_1 \tau^{-1/2} f(\tau) = 0. \tag{57}$$

The exact solution to this reduced Eq. (57) is given by:

$$f(\tau) = \sqrt{\tau}\left[c_1 J_{2/3}\left(\frac{4}{3}\sqrt{C_1}\tau^{3/4}\right) + c_2 Y_{2/3}\left(\frac{4}{3}\sqrt{C_1}\tau^{3/4}\right)\right], \tag{58}$$

where c_1 and c_2 are arbitrary integration constants, $J_\nu(\cdot)$ and $Y_\nu(\cdot)$ denote the Bessel functions of the first and second kind of order $\nu = 2/3$, respectively.

To satisfy the specific initial conditions $\Phi_1(0) = s_{1,0}$ and $\Phi_2(0) = s_{2,0}$, the exact trajectories of the auxiliary variables $u(\tau)$, $v(\tau)$, and $w(\tau)$ must be determined. Because these newly introduced variables are defined only up to a common scaling factor resulting from the quotient transformation (43), the initial scale of the denominator $w(\tau)$ can be fixed arbitrarily. Setting

$$w(0) = 1, \tag{59}$$

yields $u(0) = s_{1,0}$ and $v(0) = s_{2,0}$. The initial derivative $w'(0)$ is then calculated by evaluating (45) at $\tau = 0$, which results in:

$$w'(0) = -a_1 w(0) - a_2 u(0) - a_3 v(0) = -\left(a_1 + a_2 s_{1,0} + a_3 s_{2,0}\right). \tag{60}$$

The integration constants c_1 and c_2 are determined by matching the analytical solution $w(\tau)$ (53) to the established initial conditions (59)–(60). This requires evaluating the asymptotic behavior of the auxiliary function $f(\tau)$ as τ approaches zero. Utilizing the definition of the Bessel function argument $\xi = \frac{4}{3}\sqrt{C_1}\tau^{3/4}$ in (58) as well as the standard identity relating Bessel functions of the first and second kind, the function $Y_{2/3}(\xi)$ can be expanded as $-\frac{1}{\sqrt{3}}J_{2/3}(\xi) - \frac{2}{\sqrt{3}}J_{-2/3}(\xi)$. Substituting this expansion into (58) yields:

$$f(\tau) = \sqrt{\tau}\left[\left(c_1 - \frac{c_2}{\sqrt{3}}\right)J_{2/3}(\xi) - \frac{2c_2}{\sqrt{3}}J_{-2/3}(\xi)\right]. \tag{61}$$

Applying the standard small-argument approximations for the Bessel functions gives:

$$\sqrt{\tau}J_{2/3}(\xi) \sim \frac{(\frac{2}{3}\sqrt{C_1})^{2/3}}{\Gamma(5/3)}\tau \equiv K_J \tau, \tag{62}$$

$$\sqrt{\tau}J_{-2/3}(\xi) \sim \frac{1}{(\frac{2}{3}\sqrt{C_1})^{2/3}\Gamma(1/3)} \equiv K_{-J}. \tag{63}$$

Thus, near the origin $\tau = 0$, the function $f(\tau)$ can be approximated by the linear relation:

$$f(\tau) \approx \left(-\frac{2c_2}{\sqrt{3}}K_{-J}\right) + \left(c_1 - \frac{c_2}{\sqrt{3}}\right)K_J \tau. \tag{64}$$

Let $K_Y = -\frac{2}{\sqrt{3}}K_{-J}$. The auxiliary variable $w(\tau)$ near $\tau = 0$ is:

$$w(\tau) = e^{-\frac{a_1}{2}\tau} f(\tau) \approx \left(1 - \frac{a_1}{2}\tau\right)\left(c_2 K_Y + f'(0)\tau\right) \approx c_2 K_Y + \left(f'(0) - \frac{a_1}{2}c_2 K_Y\right)\tau. \tag{65}$$

Equating the constant term of this expansion to $w(0) = 1$ uniquely determines the second integration constant as:

$$c_2 = \frac{1}{K_Y} = -\frac{\pi}{\Gamma(2/3)}\left(\frac{2}{3}\sqrt{C_1}\right)^{2/3}. \tag{66}$$

Subsequently, equating the linear coefficient to $w'(0)$ from (60) and rearranging the obtained relation, yields the first integration constant:

$$c_1 = \frac{1}{K_J}\left[-\left(\frac{a_1}{2} + a_2 s_{1,0} + a_3 s_{2,0}\right)\right] + \frac{c_2}{\sqrt{3}}. \tag{67}$$

With the auxiliary function $f(\tau)$ and the corresponding denominator $w(\tau)$ fully defined, the numerators $u(\tau)$ and $v(\tau)$ of the transformation (5) are obtained by integrating the following first-order linear ODEs:

$$u'(\tau) = \left(a_0 + \frac{\gamma_{1,1}}{2\sqrt{\tau}}\right)w(\tau), \quad u(0) = s_{1,0}; \tag{68}$$

$$v'(\tau) = \left(b_0 + \frac{\gamma_{2,1}}{2\sqrt{\tau}}\right)w(\tau), \quad v(0) = s_{2,0}, \tag{69}$$

which yields:

$$u(\tau) = s_{1,0} + \int_0^\tau \left(a_0 + \frac{\gamma_{1,1}}{2\sqrt{s}} \right) e^{-\frac{a_1}{2}s} f(s)\,\mathrm{d}s, \tag{70}$$

$$v(\tau) = s_{2,0} + \int_0^\tau \left(b_0 + \frac{\gamma_{2,1}}{2\sqrt{s}} \right) e^{-\frac{a_1}{2}s} f(s)\,\mathrm{d}s. \tag{71}$$

The exact closed-form solutions to the original reduced ODE system (40) are then constructed by taking the ratios of the respective auxiliary functions and substituting $\tau = t^2$, which produces the final integral representations:

$$\Phi_1(t) = \frac{s_{1,0} + \int_0^{t^2} \left(a_0 + \frac{\gamma_{1,1}}{2\sqrt{s}} \right) e^{-\frac{a_1}{2}s} f(s)\,\mathrm{d}s}{e^{-\frac{a_1}{2}t^2} f(t^2)}, \tag{72}$$

$$\Phi_2(t) = \frac{s_{2,0} + \int_0^{t^2} \left(b_0 + \frac{\gamma_{2,1}}{2\sqrt{s}} \right) e^{-\frac{a_1}{2}s} f(s)\,\mathrm{d}s}{e^{-\frac{a_1}{2}t^2} f(t^2)}. \tag{73}$$

Theorem 4.1. *The system of multiplicatively coupled Caputo fractional Riccati equations of order $n = 2$, given by*

$$\begin{cases} \left({}^C\mathbf{D}^{(1/2)}\right)^2 y_1(x) = a_0 + a_1 y_1 + a_2 y_1^2 + a_3 y_1 y_2, \\ \left({}^C\mathbf{D}^{(1/2)}\right)^2 y_2(x) = b_0 + b_1 y_2 + b_2 y_2^2 + b_3 y_1 y_2, \end{cases} \tag{74}$$

subject to the initial conditions $y_i(0) = s_{i,0}$ and ${}^C\mathbf{D}^{(1/2)} y_i(0) = s_{i,1}$ for $i = 1, 2$, admits an exact analytical solution if the system parameters satisfy the constraints $a_1 = b_1$, $a_2 = b_3$, $a_3 = b_2$, and $a_2 a_0 + a_3 b_0 = a_1^2/4$. Under these conditions, the exact solutions are expressed as $y_1(x) = \Phi_1(\sqrt{x})$ and $y_2(x) = \Phi_2(\sqrt{x})$, where the generating functions Φ_1 and Φ_2 are defined by (72) and (73).

Proof. The proof follows directly from the derivations presented in this subsection. For clarity, Fig. 3 provides a schematic diagram of the required transformations and constraints used to yield the final analytical solution. □

The exact solution derived in this section is presented in terms of quadratures, meaning the problem has been reduced entirely to the evaluation of specific definite integrals. It is worth noting that integrals involving the product of exponential and Bessel functions with fractional arguments generally do not possess closed-form representations in terms of standard elementary functions. Consequently, while the analytical structure is fully resolved, the practical evaluation of the functions $\Phi_1(t)$ and $\Phi_2(t)$ for specific time points relies on numerical integration methods.

4.2.1. Computational experiments

To illustrate the proposed reduction technique and demonstrate the practical evaluation of the derived analytical solutions, a specific instance of the multiplicatively coupled Caputo fractional Riccati system (39) is considered:

$$\begin{cases} \left({}^C\mathbf{D}^{(1/2)}\right)^2 y_1(x) = 0.5 - 0.5 y_1 - y_1^2 - 0.5 y_1 y_2, \\ \left({}^C\mathbf{D}^{(1/2)}\right)^2 y_2(x) = -1.125 - 0.5 y_2 - 0.5 y_2^2 - y_1 y_2, \end{cases} \tag{75}$$

subject to

$$y_1(0) = 0.8,\ {}^C\mathbf{D}^{(1/2)} y_1(0) = -0.2,\ y_2(0) = 0.4,\ {}^C\mathbf{D}^{(1/2)} y_2(0) = 0.1.$$

Note that the system parameters are selected to satisfy the constraints identified in Theorem 4.1.

To visually illustrate the derived analytical framework and provide a comparative baseline, the solutions to the system (75) are evaluated using two distinct approaches. It is important to emphasize that the primary contribution of this work is the exact analytical derivation, therefore, this computational experiment is intended purely for visual confirmation rather than rigorous numerical benchmarking. Firstly, the exact solution derived in the previous section is evaluated via high-precision numerical quadrature of the exact integral representations. Secondly, a direct numerical approximation is computed by reformulating the original FDE system (75) as a set of four coupled FDEs of order 0.5.

It is worth noting that, in general, solving multiplicatively coupled Riccati FDE systems of order 0.5 presents significant computational challenges, which arise primarily from non-linearity, varying dynamic stiffness, and the slowly decaying non-local memory intrinsic to the Caputo operator. To address these challenges, several advanced numerical algorithms are recommended, with the optimal selection depending heavily on the system's matrix dimensionality and the physical characteristics of the underlying model. For instance, convolution quadrature-based fractional backward differentiation methods [45] are optimally utilized for extremely stiff anomalous diffusion systems with high-frequency transients that require adaptive step-sizing. Alternatively, spectral collocation methods are highly suited for optimal control trajectories requiring extremely high exponential precision and continuous smooth solutions [46].

For the purposes of the specific computational experiment presented here, however, the dynamics are characteristic of general-purpose biological modeling, representing a moderately stiff system where explicit handling of the nonlinear terms suffices.

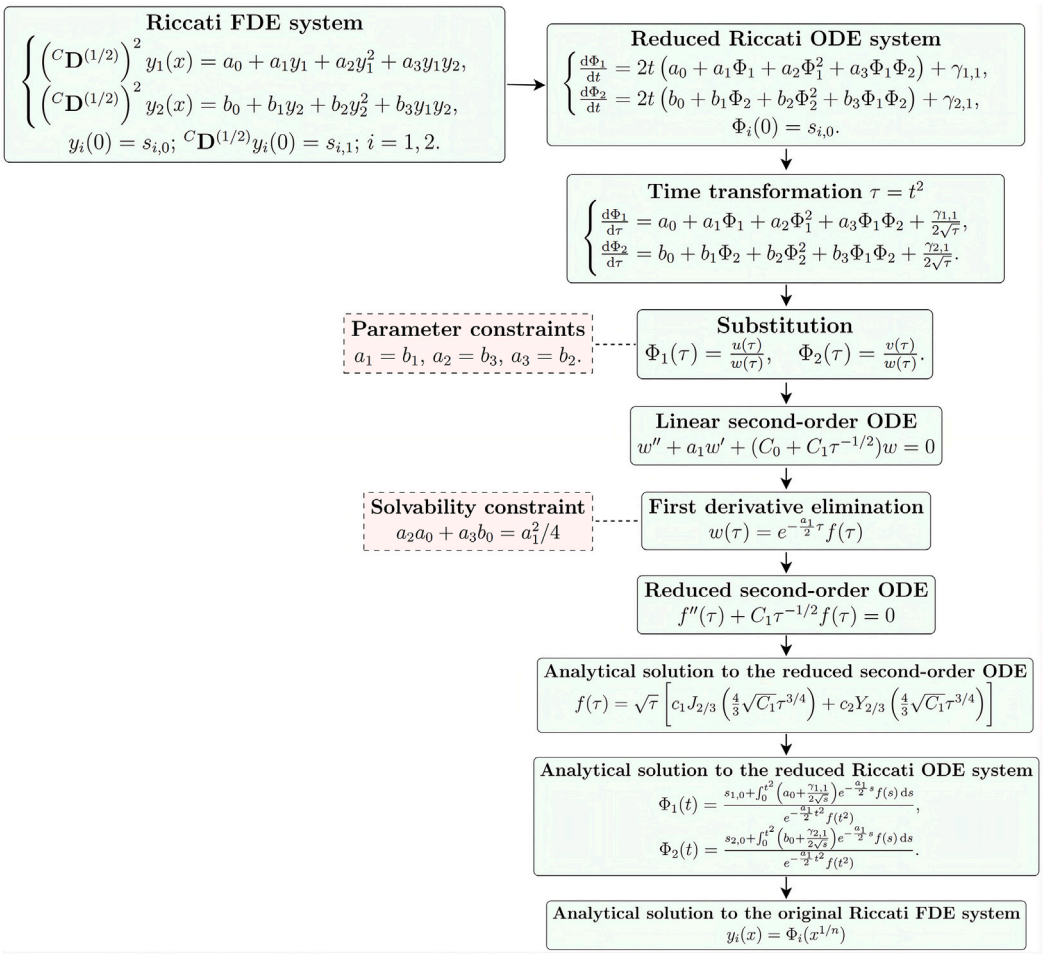

Fig. 3. Schematic diagram of the analytical solution derivation for the fractional Riccati system.

Therefore, the system was solved using the implicit product-integration rule of the trapezoidal type proposed by Garrappa [47]. This algorithm provides a convenient and robust computational baseline for this class of problems, effectively managing the fractional memory by utilizing a fast Fourier transform.

The results of this computational experiment are presented in Fig. 4. The left panel displays the generating functions $\boldsymbol{\Phi}_1(t)$ and $\boldsymbol{\Phi}_2(t)$ in the auxiliary time domain t. The right panel shows the solutions $y_1(x)$ and $y_2(x)$ to the original system in the domain x. The solid lines represent the analytical solution, while the markers indicate the solution computed by the direct numerical solver. The comparison in the right panel confirms an agreement between the evaluated exact solution and the numerical approximation, thereby illustrating the practical computability of the derived analytical framework.

5. Conclusions

The primary theoretical contribution of this study is the generalization of algebraic reduction techniques for solving FDEs, extending the framework from scalar equations to systems of nonlinear Caputo FDEs. By utilizing the algebraic properties of fractional power series, the proposed framework maps the non-local fractional operators into a local integer-order domain, effectively reducing fractional systems to equivalent systems of first-order ordinary differential equations. A key advantage of this transformation is its ability to transfer the infinite memory effects, which traditionally cause significant computational overhead, into a deterministic, time-dependent polynomial forcing term within the reduced integer-order system. This reduction bypasses the integration difficulties inherent to fractional calculus and enables the application of robust mathematical techniques to fractional dynamics.

The physical applicability of this reduction technique is demonstrated through the analysis of a multiplicatively coupled fractional Riccati system. The investigation focuses on the impact of fractional memory on parameter configurations that support solitary wave solutions in standard integer-order dynamics. Computational experiments demonstrate that the introduction of non-zero fractional

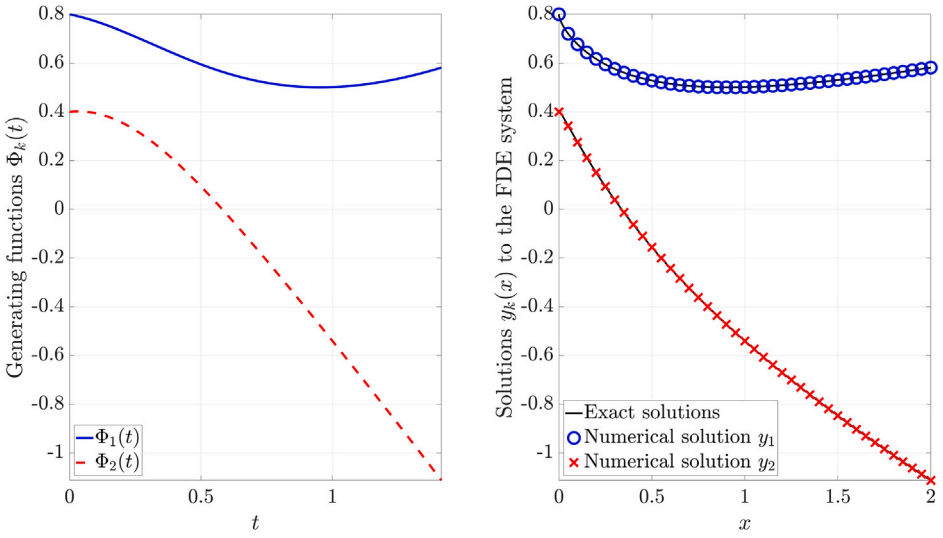

Fig. 4. Computational results for the fractional Riccati system (75). Solutions $\Phi_1(t)$ and $\Phi_2(t)$ to the reduced ODE system (40) are depicted in the left panel. The comparison of the solutions $y_1(x)$ and $y_2(x)$ to the original FDE system (75) obtained via the analytical method presented in this paper (solid lines) and a direct numerical FDE solver (markers) is presented in the right panel.

initial conditions acts as a continuous shape modulator. This memory-induced forcing fundamentally alters the amplitude, width, and asymptotic decay rates of the solitary waves, breaking the exact spatial and temporal symmetries of the standard localized behavior.

Despite the memory-induced deformation and the resulting loss of general integrability, novel exact analytical solutions for the fractional Riccati system are constructed in this study. By identifying specific parameter constraints that compensate for the non-autonomous memory effects, the nonlinear ordinary differential equation system is mapped into a higher-dimensional linear space and resolved using Bessel functions. The resulting closed-form analytical solutions, expressed as exact integral representations are validated against direct numerical approximation methods. The agreement between the analytical evaluations and the numerical simulations confirms the accuracy of the proposed reduction technique and its practical utility for analyzing complex fractional dynamics. Ultimately, the ability to analytically resolve these memory-induced deformations provides a rigorous mathematical foundation for understanding anomalous transport, viscoelastic damping, and non-local wave interactions across a wide spectrum of applications in fractional mechanics and physics.

Regarding the broad applicability of this methodology for obtaining closed-form solutions in general physical models, it is crucial to distinguish between the algebraic reduction process and the classical integrability of the resulting system. The proposed mapping from the fractional domain to the equivalent ODE system is highly general and can be broadly applied to any coupled system of Caputo FDEs governed by the $\left({}^{C}\mathbf{D}^{(1/n)}\right)^{n}$ operator, provided the nonlinear terms are analytic within the fractional power series algebra. Once the fractional system is successfully reduced to the integer-order domain, the existence of a final closed-form analytical solution depends entirely on the classical integrability of the resulting ODEs. Consequently, this methodology allows researchers to apply the vast, established set of exact ODE techniques (such as Lie symmetry analysis, integrating factors, and variable transformations) to solve fractional systems. Furthermore, even in scenarios where the resulting ODEs exhibit non-integrable dynamics that preclude a closed-form solution, this algebraic reduction remains highly advantageous, as it entirely eliminates the computational bottleneck of infinite fractional memory, allowing for the application of standard, highly optimized integer-order numerical solvers.

Looking forward, a promising direction for future research involves generalizing this algebraic reduction methodology to a broader class of fractional operators. For instance, adapting the framework to Riemann–Liouville derivatives would necessitate a structural modification of the underlying fractional power series algebra to natively accommodate the distinct singular behavior and integral-type initial conditions characteristic of the Riemann–Liouville formulation. Furthermore, extending the proposed mapping techniques to the broader class of general fractional derivatives featuring Sonine kernels [48–50] would require constructing new algebraic isomorphisms bounded by generalized convolution kernels rather than classical power laws. Implementing such generalizations will further expand the applicability of exact algebraic reduction techniques to an even wider array of complex, non-local physical systems.

CRediT authorship contribution statement

Inga Telksnienė: Writing – review & editing, Writing – original draft, Visualization, Validation, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization. **Tadas Telksnys:** Methodology, Formal analysis, Conceptualization. **Romas Marcinkevičius:** Validation, Software. **Zenonas Navickas:** Methodology, Formal analysis. **Raimondas Čiegis:** Validation, Supervision, Investigation. **Minvydas Ragulskis:** Writing – review & editing, Supervision, Methodology.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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