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Applications of Generalized Beta-distribution in Quality Control Models

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Introduction

Models of multistage continuous control of mechatronic products are analyzed in this work similarly as in publication [1], when production classification errors of the first and second types are present [2, 3, 4]. Main difference is that the area of analyzed structural variants of control process will be broadened and more general beta-distribution will be applied as well. The main distinguishing of this distribution is that the random value (r.v.) can vary in any interval (including from 0 to 1 also) and in such manner this generalized double-parameter distribution may be applied not only in quality control for description of the defectivity level, but also to solve various different engineering problems using stochastic methods. We will discuss how generalized beta-distribution is integrated into the general space of stochastic distributions, when they are selected for particular applications.

The selection of stochastic distributions

In the engineering practice during the experiments the empiric results of the observations are obtained, according to which it is possible to calculate some certain numerical characteristics of the observed r.v. X. According to these characteristics we need to select the stochastic model which would describe the distribution of r.v. X. For this reason we require the sufficiently broad set (family) of stochastic distributions and some certain rule, on the basis of which it would be possible to select most suitable distribution from the available set for particular case.

Distribution families offered by Johnson and Pearson [5, 6] are applicable most widely. We will concentrate on the family of Pearson curves, since the generalized beta-distribution is one of the main in this family. In general case the density f(x) of r.v. X belongs to the family of Pearson curves, if it meets the differential equation (1)

$$\frac{1}{f(x)}\frac{df(x)}{dx} = \frac{d\ln f(x)}{dx} = \frac{x-b_1}{b_2x^2+b_1x+b_0}, \tag{1}$$

where parameters b_0, b_1, b_2 – the real numbers.

The shape of the density depends on the roots of polynomial $b_2x^2+b_1x+b_0$. When there are two roots of opposite signs, we have the distribution of the I type with density

$$f(x) = C(x-v_0)^{a-1}(v_1-x)^{b-1},$$
$$v_0 \le x \le v_1,\ a>0,\ b>0; \tag{2}$$

where C – normalizing multiplier (constant).

This is the generalized beta-distribution [6] and the random value $Y=(X-v_0)/(v_1-v_0)$ will already be distributed according to beta distribution with parameters a, b and $0 \le y \le 1$.

There are seven types of distributions in the set of Pearson curves (including such distributions as beta, gamma, chi-square, Fisher, Student and Gaussian) [6], although in [7] this classification was extended up to 12 types. Sets of I, IV and VII distributions are the widest (according to [6] classification). It is considered in probability theory, that gamma, beta (generalized) and Gaussian distributions are the main ones.

Each distribution of probabilities fully characterizes using k-th order initial $\alpha_{(k)}$ or central $\mu_{(k)}$ moments:

$$\left.\begin{aligned}\alpha_{(k)} &= EX^k = \int_{-\infty}^{\infty} x^k f(x)dx,\ \mathrm{k}=1,2,\ldots\\ \mu_{(k)} &= E(X-EX)^k = \int_{-\infty}^{\infty}(x-EX)^k f(x)dx,\ \mathrm{k}=1,2,\ldots\end{aligned}\right\}; \tag{3}$$

where EX – mathematical mean or average. In practice it is sufficient to have the first four (k=1, 2, 3, 4) initial moments, since central moments can be expressed using initial moments ($\mu_{(1)} = 0$ always):

$$\left.\begin{aligned} &\mu_{(2)} = \alpha_{(2)} - \alpha_{(1)}^2, \mu_{(3)} = \alpha_{(3)} - \alpha_{(2)}\alpha_{(1)} + 2\alpha_{(1)}^3, \\ &\mu_{(4)} = \alpha_{(4)} - 4\alpha_{(3)}\alpha_{(1)} + 6\alpha_{(2)}\alpha_{(1)}^2 - 3\alpha_{(1)}^4 \end{aligned}\right\}; \quad (4)$$

where $\alpha_{(1)} \equiv EX = \mu$, $\mu_{(2)} \equiv VX = \delta^2$ [8].

The asymmetry factor γ_A and excess γ_E are equal to [6]

$$\gamma_A = \frac{\mu_{(3)}}{\delta^3} = \frac{\mu_{(3)}}{\mu_{(2)}^{3/2}}; \; \gamma_E = \frac{\mu_{(4)}}{\delta^4} - 3 = \frac{\mu_{(4)}}{\mu_{(2)}^2} - 3. \quad (5)$$

Note. For Gaussian distribution $\gamma_A = \gamma_E = 0$ i.e. $\frac{\mu_{(4)}}{\mu_{(2)}^2} = 3$.

For the selection of the distribution from Johnson or Pearson sets of curves the parameters β_A and β_E are used [6], which by some certain meaning describe the shape of the distribution:

$$\beta_A = \gamma_A^2 = \frac{\mu_{(3)}^2}{\mu_{(2)}^3}, \; \beta_E = \gamma_E + 3 = \frac{\mu_{(4)}}{\mu_{(2)}^2}. \quad (6)$$

The following relation formulas are valid for the family of Pearson curves (1):

$$b_0 = \frac{2+\delta}{2(1+2\delta)}, \; b_1 = \frac{\delta\sqrt{\beta_A}}{2(1+2\delta)}, \; b_2 = \frac{\delta}{2(1+2\delta)}, \quad (7)$$

where $\delta = \frac{2\beta_E - 3\beta_A - 6}{\beta_E + 3}$.

Each distribution (considering its shape) can be visualized by points on the plane $\beta_A 0 \beta_E$. The distribution for which (β_A, β_E) acquire the single value is represented by point (for example, for the Gaussian distribution (N): (β_A, β_E)=(0, 3)). Distribution, which has one shape parameter, is represented by particular curve, and with two shape parameters – particular area of the plane $\beta_A 0 \beta_E$ (Fig. 1).

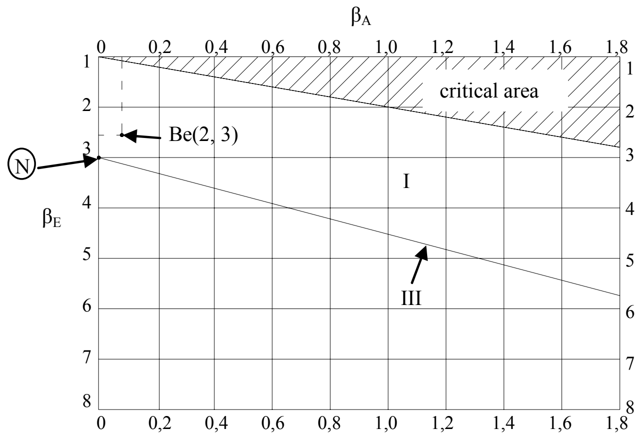

Fig. 1. The family of Pearson curve (types I and III)

The family of distributions of the III type is shown in Fig. 1 as straight line, and of the I type (generalized beta-distribution) – as area between the III type and the line $\beta_E - \beta_A - 1 = 0$, which defines the critical area in which distributions do not exist. Most of other-type Pearson distributions fall into the area below curve III (see Fig. 10 [6]).

In general case, when moments of the random value are known, the type of the curve is selected according to β_A, β_E values, then distribution parameters are expressed using moments and so the density f(x) is completely characterized.

Let's illustrate it by formal example. Assume, that we want to visualize the particular beta-distribution X~Be(2, 3) in the plane $\beta_A 0 \beta_E$, when $a = 2$, $b = 3$, $v_0 = 0 \le x \le v_1 = 1$. The initial moments are

$$\alpha_{(k)} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 x^{a+k-1}(1-x)^{b-1}dx = \frac{\Gamma(a+b)\Gamma(a+k)}{\Gamma(a)\Gamma(a+b+k)} =$$
$$= \frac{a+k-1}{a+b+k-1}\alpha_{(k-1)}, \text{ k=2, 3, 4; } \alpha_{(1)} = \frac{a}{a+b}; \quad (8)$$

where $\Gamma(z)$ – gamma function [3, 4].

According to (4), (6), (8) we have – $\alpha_{(1)} = \mu = 0{,}4$; $\alpha_{(2)} = 0{,}2$; $\alpha_{(3)} = \frac{4}{35}$; $\alpha_{(4)} = \frac{1}{14}$; $\alpha_{(3)} = \frac{4}{35}$; $\mu_{(2)} = \delta^2 = 0{,}04$; $\mu_{(3)} = \frac{2}{875}$; $\mu_{(4)} = \frac{33}{8750}$ and $\beta_A = \frac{4}{49} = 0{,}082$, $\beta_E = \frac{33}{14} = 2{,}357$. We place the point (β_A, β_E) = (0,082, 2,357) on the plane $\beta_A 0 \beta_E$ (Fig. 1) and it is obvious, that this is the I type distribution of Pearson family.

Generalized beta-distribution

When the random value Y~Be(a, b), we have the density of beta-distribution

$$\varphi(y \mid a,b) = \frac{1}{B(a,b)} y^{a-1}(1-y)^{b-1}, \; 0 \le y \le 1; \quad (9)$$

where $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ – beta-function, with numerical characteristics of the random value Y:

$$EY = \mu_y = \frac{a}{a+b}, \; VY = \delta_y^2 = \frac{\mu_y(1-\mu_y)}{a+b+1}, \; \delta_y = \sqrt{VY}. \quad (10)$$

We perform transformation $X = Y(v_1 - v_0) + v_0$. Then r.v. X is distributed according to generalized beta-law with density

$$f(x) = f(x \mid a,b) = |y'(x)| \varphi[y(x)] =$$
$$= C(x - v_0)^{a-1}(v_1 - x)^{b-1}, \; v_0 \le x \le v_1, \quad (11)$$

where $y(x)=\dfrac{x-v_0}{v_1-v_0}$, $y'(x)=\dfrac{\partial y(x)}{\partial x}=\dfrac{1}{v_1-v_0}$,

$$\varphi[y(x)]=\frac{1}{B(a,b)(v_1-v_0)^{a+b-2}}(x-v_0)^{a-1}(v_1-x)^{b-1},$$

$$C=\frac{1}{B(a,b)(v_1-v_0)^{a+b-1}}=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)(v_1-v_0)^{a+b-1}}=const.$$

The numerical characteristics of the random value X:

$$EX=\mu=v_0+(v_1-v_0)\mu_y=\frac{av_1+bv_0}{a+b}, \quad (12)$$

$$VX=\delta^2=(v_1-v_0)^2\delta_y^2=\frac{ab}{a+b+1}\left(\frac{v_1-v_0}{a+b}\right)^2,$$

$$\delta=\sqrt{VX}. \quad (13)$$

When a=b=1, we have the uniform distribution in interval (v_0,v_1): $X\sim R(v_0,v_1)$ with density

$$f(x\,|\,1;1)=\frac{1}{v_1-v_0},\ v_0\le x\le v_1,\ \text{a=b=1}. \quad (14)$$

From (12), (13) equations we obtain a and b expressions

$$a=\frac{\mu-v_0}{v_1-v_0}\left[\frac{(\mu-v_0)(v_1-\mu)}{\delta^2}-1\right],\ b=\frac{v_1-\mu}{v_1-v_0}a. \quad (15)$$

After substituting the estimates $\hat{\eta}=\bar{x}$, $\hat{\delta}^2=s^2$ into (15) [3, 4] and having the a priori information about the values of v_0, v_1, we obtain the estimates $\hat{a},\hat{b}$ and in such manner we select particular density according to empirical data.

Transformations of densities in continuous control

● Let's analyze two-stage continuous control (Fig. 2). Analogous was discussed in [1] Fig. 2, but in the current case the probability density $g(\theta)$ of defective product between stages K_1 and K_2 is described using the density of generalized beta-distribution.

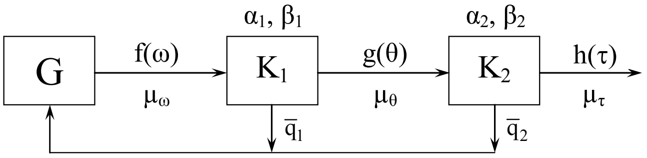

Fig. 2. The diagram of two-stage control

We have

$$g(\theta)=C(\theta-v_0)^{a-1}(v_1-\theta)^{b-1},\ 0\le v_0\le\theta\le v_1\le 1; \quad (16)$$

where the normalizing factor C is found according to (11). After direct transformation **T** (see [1]) $g(\theta)$ is transformed into density $h(\tau)$

$$h(\tau)=|\theta'(\tau)|\,g[\theta(\tau)]=C_\tau\frac{(\tau-\tau_0)^{a-1}(\tau_1-\tau)^{b-1}}{(1+C_2\tau)^{a+b}},$$

$$\tau_0\le\tau\le\tau_1; \quad (17)$$

where $C_\tau=\dfrac{C}{\widetilde{\beta}_2^{a+b-1}(1+C_2\tau_0)^{a-1}(1+C_2\tau_1)^{b-1}}=const.$,

$$\theta(\tau)=\frac{\tau}{\widetilde{\beta}_2(1+C_2\tau)}=\frac{\tau}{\widetilde{\beta}_2+\widetilde{\gamma}_2\tau},\ \theta'(\tau)=\frac{1}{\widetilde{\beta}_2(1+C_2\tau)^2},$$

$$g[\theta(\tau)]=\frac{C}{\widetilde{\beta}_2^{a+b-2}}\left(\frac{\tau-\tau_0}{1+C_2\tau_0}\right)^{a-1}\left(\frac{\tau_1-\tau}{1+C_2\tau_1}\right)^{b-1}\frac{1}{(1+C_2\tau)^{a+b-2}},$$

$$\widetilde{\beta}_i=\frac{\beta_i}{1-\alpha_i},\ \widetilde{\gamma}_i=1-\widetilde{\beta}_i=\frac{\gamma_i}{1-\alpha_i},\ \widetilde{\gamma}_i=1-\alpha_i-\beta_i,\ \text{i=1, 2},$$

$$C_i=\frac{1}{\widetilde{\beta}_i}-1=\frac{\widetilde{\gamma}_i}{\widetilde{\beta}_i}=\frac{\gamma_i}{\beta_i};\ \tau_0=\frac{\widetilde{\beta}_2v_0}{1-\widetilde{\gamma}_2v_0},\ \tau_1=\frac{\widetilde{\beta}_2v_1}{1-\widetilde{\gamma}_2v_1}.$$

After the reverse transformation **A,** $g(\theta)$ is transformed into the density f(ω)

$$f(\omega)=C_\omega\frac{(\omega-\omega_0)^{a-1}(\omega_1-\omega)^{b-1}}{(1+\widetilde{\gamma}_1\omega)^{a+b}},\ \omega_0\le\omega\le\omega_1, \quad (18)$$

where $C_\omega=\dfrac{C\widetilde{\beta}_1^{a+b-1}}{(1-\widetilde{\gamma}_1\omega_0)^{a-1}(1-\widetilde{\gamma}_1\omega_1)^{b-1}}=const.$,

$$\omega_0=\frac{v_0}{\widetilde{\beta}_1+\widetilde{\gamma}_1v_0}=\frac{v_0}{\widetilde{\beta}_1(1+c_1v_0)},\ \omega_0=\frac{v_1}{\widetilde{\beta}_1(1+c_1v_1)}.$$

Densities f(ω) and $h(\tau)$ are named as the second-order generalized beta-distributions [7].

The mode (point of maximum) θ_M of the density $g(\theta)$ is

$$\theta_M=v_0+(v_1-v_0)\frac{a-1}{a+b-2}. \quad (19)$$

We will use the transformation X=τ-τ$_0$ for the density $h(\tau)$ and in this way we shall obtain the density h(x)

$$h(x)=C_x\frac{x^{a-1}(\Delta_\tau-x)^{b-1}}{(1+\beta_\tau x)^{a+b}},\ 0\le x\le\Delta_\tau; \quad (20)$$

where $C_x=const.$, $\Delta_\tau=\tau_1-\tau_0$, $\beta_\tau=\dfrac{C_2}{1+C_2\tau_0}$;

Assuming that the first derivative h'(x)=0, we obtain the equation of extremes (21)

$$x^{a-2}(\Delta_\tau-x)^{b-2}[2\beta_\tau x^2-\lambda_\tau x+(a-1)\Delta_\tau]=0; \quad (21)$$

where $\lambda_\tau=(b+1)\Delta_\tau\beta_\tau+a+b-2$

After solving (21), we have that the mode τ_M equals

$$\tau_M = \tau_0 + x_M = \tau_0 + \frac{\lambda_\tau}{4B_\tau}[1 \pm \sqrt{1 - \frac{8(a-1)\lambda_\tau B_\tau}{\lambda_\tau^2}}]. \quad (22)$$

Analogously, after making the substitution Z=ω-ω₀, we have

$$\omega_M = \omega_0 + z_M = \omega_0 + \frac{\lambda_\omega}{4B_\omega}[1 \pm \sqrt{1 - \frac{8(a-1)\lambda_\omega B_\omega}{\lambda_\omega^2}}]. \quad (23)$$

where $\Delta_\omega = \omega_1 - \omega_0$, $B_\omega = -\frac{\tilde{\gamma}_1}{1 - \tilde{\gamma}_1 \omega_0}$,

$\lambda_\omega = (b+1)\Delta_\omega B_\omega + a + b - 2$.

Averages μ_τ and μ_ω can be calculated by (24):

$$\mu_\tau = \int_{\tau_0}^{\tau_1} \tau h(\tau) d\tau, \ \mu_\omega = \int_{\omega_0}^{\omega_1} \omega f(\omega) d\omega. \quad (24)$$

In practice it is more convenient to use the functions $\tau(\theta)$, $\omega(\theta)$ and the density $g(\theta)$:

$$\mu_\tau = \int_{v_0}^{v_1} \tau(\theta) g(\theta) d\theta, \ \mu_\omega = \int_{v_0}^{v_1} \omega(\theta) g(\theta) d\theta; \quad (25)$$

where $\tau(\theta) = \frac{\tilde{\beta}_2 \theta}{1 - \tilde{\gamma}_2 \theta}$, $\omega(\theta) = \frac{\theta}{\tilde{\beta}_1 (1 + c_1 \theta)} = \frac{\theta}{\tilde{\beta}_1 + \tilde{\gamma}_1 \theta}$.

The averages of returning flows according to [3, 4] are equal to

$$\bar{q}_1 = \alpha_1 + \gamma_1 \mu_\omega, \ \bar{q}_2 = \alpha_2 + \gamma_2 \mu_\theta. \quad (26)$$

According to (16), (25) we obtain for **A** transformation:

$$\mu_\omega = \frac{C}{\tilde{\beta}_1} \int_{v_0}^{v_1} \frac{\theta(\theta - v_0)^{a-1}(v_1 - \theta)^{b-1}}{1 + c_1 \theta} d\theta = \frac{C}{\tilde{\beta}_1} I =$$

$$= \frac{1}{\tilde{\gamma}_1}[1 - C I_\omega], \quad (27)$$

where

$$I = \frac{1}{c_1} \int_{v_0}^{v_1} 1 - \frac{1}{1 + c_1 \theta} (\theta - v_0)^{a-1} (v_1 - \theta)^{b-1} d\theta = \frac{1}{c_1}(I_1 - I_\omega),$$

$$I_1 = \int_{v_0}^{v_1} (\theta - v_0)^{a-1}(v_1 - \theta)^{b-1} d\theta = (v_1 - v_0)^{a+b-1} B(a,b) =$$

$$= \frac{1}{C},$$

$$I_\omega = \int_{v_0}^{v_1} \frac{(\theta - v_0)^{a-1}(v_1 - \theta)^{b-1}}{1 + c_1 \theta} d\theta.$$

Analogously for the direct transformation T we receive the following:

$$\mu_\tau = \frac{1}{c_2}(C I_\tau - 1), \quad (28)$$

where $I_\tau = \int_{v_0}^{v_1} \frac{(\theta - v_0)^{a-1}(v_1 - \theta)^{b-1}}{1 - \tilde{\gamma}_2 \theta} d\theta$.

Similarly like in [1] we assume, that it is sufficient to use the whole-number values of parameters a and b (a≥1, b≥1) during the modeling. After integration of I_ω from (27) and I_τ from (28), we receive the following expressions:

$$I_\omega = \frac{1}{c_1^{a+b-1}} \left\{ (-1)^{a-1} x_0^{a-1} [x_1^{b-1} \ln \frac{x_1}{x_0} + \right.$$

$$+ \sum_{i=1}^{b-1} C_{b-1}^i (-1)^i x_1^{b-i-1} \frac{x_1^i - x_0^i}{i}] + \sum_{j=1}^{a-1} C_{a-1}^j (-1)^{a-j-1} x_0^{a-j-1} \cdot$$

$$\left. \cdot \sum_{i=0}^{b-1} C_{b-1}^i (-1)^i x_1^{b-i-1} \frac{x_1^{i+j} - x_0^{i+j}}{i+j} \right\}; \quad (29)$$

where $x_0 = 1 + c_1 v_0$, $x_1 = 1 + c_1 v_1$; $C_n^m = \frac{n!}{m!(n-m)!}$;

$$I_\tau = \frac{(-1)^{a+b-1}}{\tilde{\gamma}_2^{a+b-1}} \left\{ (-1)^{a-1} y_0^{a-1} + [\sum_{i=1}^{b-1} C_{b-1}^i (-1)^{i+1} y_1^{b-i-1} \frac{y_0^i - y_1^i}{i} - \right.$$

$$- y_1^{b-1} \ln \frac{y_0}{y_1}] + \sum_{j=1}^{a-1} C_{a-1}^j (-1)^{a-j-1} y_0^{a-j-1} \cdot$$

$$\left. \cdot \sum_{i=0}^{b-1} C_{b-1}^i (-1)^{i+1} y_1^{b+i-1} \frac{y_0^{i+j} - y_1^{i+j}}{i+j} \right\}; \quad (30)$$

where $y_0 = 1 - \tilde{\gamma}_2 v_0$, $x_1 = 1 - \tilde{\gamma}_2 v_1$.

Furthermore, when a and b have whole-number values, we have

$$C = \frac{(a+b-1)!}{(a-1)!(b-1)!(v_1 - v_0)^{a+b-1}}, \quad (31)$$

where a≥1, b≥1 – whole numbers.

If we consider s-stage direct transformation $\mathbf{T_s}$ (according to [1] Fig. 3), then the density $h_s(\tau_s)$ has the form of (17), when we substitute c_{1s} instead of c_2

$$c_{1s} = \frac{1}{\tilde{\beta}_{1s}} - 1, \ \tilde{\beta}_{1s} = \prod_{i=1}^{s} \tilde{\beta}_i, \ \text{i} = 1\text{-s}. \quad (32)$$

Averages μ_s are calculated analogously, when instead of c_2 we put c_{1s} into expression and instead $\tilde{\gamma}_2$ we put $\tilde{\gamma}_{1s} = 1 - \tilde{\beta}_{1s}$.

It is obvious, that all the received models with generalized beta-distribution can be immediately used for transformations of beta-distribution, after inserting values $v_0 = 0$ and $v_1 = 1$.

●● Let's consider two-stage continuous control (modified), in which the returning flow is passed into its own repair (regeneration) operation R_i after each control stage K_i, i=1, 2 (see Fig. 3).

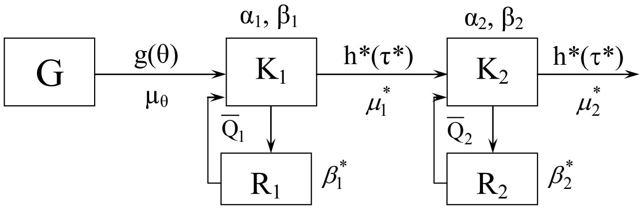

Fig. 3. The modified two-stage continuous control

In each repair operation R_i all the products rejected during the control K_i are repaired and returned back to the K_i. The cycle is repeated until all the products which have been passed into the control K_i, will be accepted as good, i.e. $\overline{p}_i = 1$ (see [1]). In average the $\overline{Q}_i$ part of products is returned to the repair operation in each stage. According to the [2] during the repair operation in such scheme the second type error $\beta_i^* = const.$, and the first type error $\alpha_i^* = 0$ (does not exist), since the assumption is made, that all the rejected products are repaired in the repair operations. Then [2]

$$\tau^* = \beta_{01}\theta\,,\ \tau_2^* = \beta_{02}\tau^* = \beta_{01}\beta_{02}\theta\,; \qquad (33)$$

where $\beta_{01} = \dfrac{\beta_1}{1-\beta_1^*(1-\beta_1)}$, $\beta_{02} = \dfrac{\beta_2}{1-\beta_2^*(1-\beta_2)}$.

Let's consider, that $g(\theta)$ (Fig. 3) is beta-density with $v_0 = 0$, $v_1 = 1$. Then after T_1^* transformation $\tau^* = \beta_{01}\theta$ we obtain already generalized beta-density h*(τ*), where $0 \le \tau^* \le \beta_{01} < 1$:

$$T_1^* : h^*(\tau^*) = |\,\theta'(\tau^*)\,|\, g[\theta(\tau^*)] =$$

$$= \frac{B^{-1}(a,b)}{\beta_{01}^{a+b-1}}(\tau^*)^{a-1}(\beta_{01}-\tau^*)^{b-1}, \qquad (34)$$

where $\theta'(\tau^*) = \dfrac{1}{\beta_{01}}$, $\theta(\tau^*) = \dfrac{\tau^*}{\beta_{01}}$, $B^{-1}(a,b) = \dfrac{1}{B\ (a,b)}$.

After the second stage K_2 we have (the transformation T_2^*):

$$T_2^* : h_2^*(\tau_2^*) = \frac{B^{-1}(a,b)}{\overline{\beta}_{12}^{a+b-1}}(\tau_2^*)^{a-1}(\overline{\beta}_{12}-\tau_2^*)^{b-1},$$

$$0 \le \tau_2^* \le \overline{\beta}_{12}\,; \qquad (35)$$

where $\overline{\beta}_{12} = \beta_{01}\beta_{02}$.

Generally we can write, that in the modified continuous control scheme with number s of the control stages K_i which are connected in series and with their own repair operations R_i, i=1-s, the density after the operation K_s is

$$h_s^*(\tau_s^*) = \frac{B^{-1}(a,b)}{\overline{\beta}_{1s}^{a+b-1}}(\tau_s^*)^{a-1}(\overline{\beta}_{1s}-\tau_s^*)^{b-1}\,,\ 0 \le \tau_s^* \le \overline{\beta}_{1s}\,, \qquad (36)$$

where $\widetilde{\beta}_{1s} = \prod_{i=1}^{s}\widetilde{\beta}_{0i}$.

The mode τ_{iM}^* of the density $h_i^*(\tau_i^*)$ is

$$\tau_{iM}^* = \overline{\beta}_{1i}\frac{a-1}{a+b-2}\,,\ \text{i=1-s}, \qquad (37)$$

and the averages μ_i^*:

$$\mu_i^* = \overline{\beta}_{1i}\frac{a}{a+b}\,. \qquad (38)$$

If $\beta_{01} = \beta_{02} = \ldots = \beta_{0s} \equiv \beta_0$, we have $\overline{\beta}_{12} = \beta_0^s$, $\tau_s^* = \beta_0^s\theta$ and

$$h_s^*(\tau_s^*) = \frac{B^{-1}(a,b)}{\beta_0^{s(a+b-1)}}(\tau_s^*)^{a-1}(\beta_0^s-\tau_s^*)^{b-1}\,,\ 0 \le \tau_s^* \le \overline{\beta}_0^s\,. \qquad (39)$$

The averages of multifold returning flows $\overline{Q}_i$ according to [3, 4] equals to

$$\overline{Q}_i = \frac{\beta_1}{1-\alpha_i}(\alpha_i + c_i\,\mu_i^*)\,, \qquad (40)$$

where c_i is found according to (17).

Note. The reverse transformation **A***: $\omega = \theta/\beta_0$ has the meaning if $v_1 \le \beta_0$, i.e. when $v_1/\beta_0 \le 1$.

Practical implementations

Example 1: The control scheme according to Fig. 2 with the error probabilities $\alpha_1 = \alpha_2 \equiv \alpha = 0{,}08$, $\beta_1 = \beta_2 \equiv \beta = 0{,}23$. Parameters of densities $a = b = 2$, when $v_0 = 0{,}1$ and $v_1 = 0{,}6$.

Modeling results:

$$\mu_\omega = \frac{1}{\widetilde{\gamma}}\left\{1 - \frac{3}{(c\Delta_\theta)^2}[x_0 - x_1(\frac{2x_0}{c\Delta_\theta}\ln\frac{x_1}{x_0} - 1)]\right\},$$

$$\mu_\tau = \frac{1}{c}\left\{1 - \frac{3}{(\widetilde{\gamma}\Delta_\theta)^2}[y_0 - y_1(\frac{2y_0}{\widetilde{\gamma}\Delta_\theta}\ln\frac{y_0}{y_1} - 1)] - 1\right\},$$

$\tilde{\beta} = \frac{\beta}{1-\alpha} = \frac{1}{4}$, $\tilde{\gamma} = 1 - \tilde{\beta} = \frac{3}{4}$, $c = \frac{1}{\tilde{\beta}} - 1 = 3$,
$\Delta_\theta = v_1 - v_0 = 0{,}5$;

$x_0 = 1 + cv_0 = 1{,}3$, $x_1 = 1 + cv_1 = 2{,}8$, $y_0 = 1 - \tilde{\gamma} v_0 = 0{,}925$,
$y_1 = 1 - \tilde{\gamma} v_1 = 0{,}55$; $\gamma = 1 - \alpha - \beta = 0{,}69$;
$g(\theta) = 48(\theta - 0{,}1)(0{,}6 - \theta)$, $\mu_\theta = \theta_M = 0{,}35$;

A: $f(\omega) = \frac{2{,}73(\omega - 0{,}3077)(0{,}8571 - \omega)}{(1 - 0{,}75\omega)^4}$, $\begin{cases} \omega_0 = 0{,}3077, \\ \omega_1 = 0{,}8571 \end{cases}$;

$\mu_\omega = 0{,}6644$, $\omega_M = 0{,}7473$, $\bar{q}_1 = 0{,}5384$;

T: $h(\tau) = \frac{1563(\tau - 0{,}027)(0{,}2727 - \tau)}{(1 + 3\tau)^4}$, $\begin{cases} \tau_0 = 0{,}027, \\ \tau_1 = 0{,}2727 \end{cases}$;

$\mu_\tau = 0{,}1247$, $\tau_M = 0{,}0939$, $\bar{q}_2 = 0{,}3215$.

We receive, that in this case the following inequalities are valid:

$\tau_0 < v_0 < \tau_1 < \omega_0 < v_1 < \omega_1 < 1$.

Densities f(ω), $g(\theta)$, $h(\tau)$ are shown in Fig. 4.

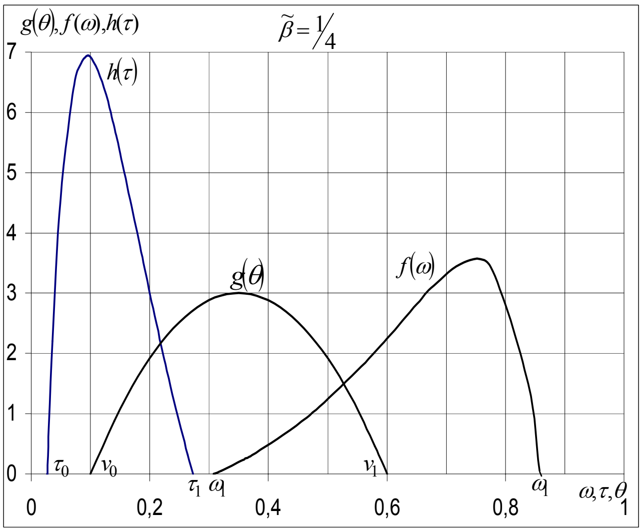

Fig. 4. Densities of defectivity levels, when a=b=2, $v_0 = 0{,}1$, $v_1 = 0{,}6$

Example 2 (two cases)

Case 2.1: The control scheme according to Fig. 3 (localized returning flows). The scheme of errors according to Fig. 3 (localized returning flows). The error probabilities $\alpha_1 = \alpha_2 \equiv \alpha = 0{,}2$, $\beta_1 = \beta_2 \equiv \beta = 0{,}4$, $\beta_1^* = \beta_2^* \equiv \beta^* = \frac{1}{3}$. Parameters a=b=1 (uniform density), $v_0 = 0$ and $v_1 = 1$.

Modeling results: $\beta_0 = 0{,}5$, $\gamma = 0{,}4$, $c = 1$, $\bar{\beta}_{12} = \beta_0^2 = \frac{1}{4}$; $g(\theta) = 1$, when $0 \le \theta \le 1$, $\mu_\theta = 0{,}5$;

$T_1^* : h^*(\tau^*) = 2$, when $0 \le \tau^* \le 0{,}5$, $\mu_1^* = 0{,}25$, $\overline{\mathrm{Q}}_1 = 0{,}5625$;

$T_2^* : h_2^*(\tau_2^*) = 4$, when $0 \le \tau_2^* \le 0{,}25$, $\mu_2^* = 0{,}125$, $\overline{\mathrm{Q}}_2 = 0{,}3906$.

Densities $g(\theta)$, $h^*(\tau^*)$, $h_2^*(\tau_2^*)$ are shown in the Fig. 5.

Case 2.2 (for comparison with the case 2.1): The control scheme according to Fig. 2 when we substitute the density $g(\theta)$ instead of the f(ω), and the density $h(\tau)$ – density $h_2(\tau_2)$ instead of the $g(\theta)$ (see [1] Fig. 3) – i.e. we apply the two-stage scheme of direct transformation T_1, T_2.

The parameters of the error probability and the density are analogous as in the case 2.1: $\alpha = 0{,}2$, $\beta = 0{,}4$, a=b=1, $v_0 = 0$, $v_1 = 1$ (the density $g(\theta)$ as in the case 2.1).

Modeling results:

$\tilde{\beta} = \tilde{\gamma} = 0{,}5$, $\tilde{\beta}_{12} = \tilde{\beta}^2 = \frac{1}{4}$, $\tilde{c} = \frac{1}{\tilde{\beta}^2} - 1 = 3$;

$T_1 : h(\tau) = \frac{1}{\tilde{\beta}(1 + c\tau)^2} = \frac{2}{(1+\tau)^2}$, $0 \le \tau \le 1$,
$\bar{q}_1 = \alpha + \gamma\mu_\theta = 0{,}4$;

$\mu_1 = \frac{1}{c}(\frac{1}{\tilde{\gamma}} \ln \frac{1}{\tilde{\beta}} - 1) = 0{,}3863$, $h(0) = 2$, $h(1) = 0{,}5$;

$T_2 : h_2(\tau_2) = \frac{1}{\tilde{\beta}^2 (1 + \tilde{c}\,\tau_2)^2} = \frac{4}{(1 + 3\tau_2)^2}$, $0 \le \tau_2 \le 1$,
$\bar{q}_2 = \alpha + \gamma\mu_1 = 0{,}3$;

$\mu_2 = \frac{1}{\tilde{c}}(\frac{1}{1 - \tilde{\beta}^2} \ln \frac{1}{\tilde{\beta}^2} - 1) = 0{,}2828$, $h_2(0) = 4$,
$h_2(1) = 0{,}25$.

Densities $h(\tau)$ and $h_2(\tau_2)$ are shown in Fig. 5.

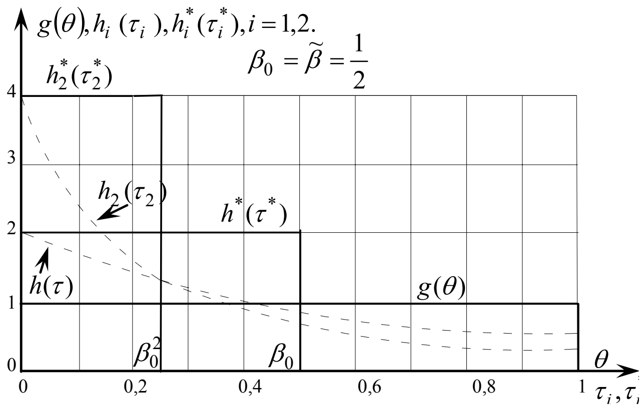

Fig. 5. Densities of defectivity levels, when a=b=1, $v_0 = 0$, $v_1 = 1$

It is obvious, that according to the values of the defectivity level average μ_i and μ_i^* the transformation T_1^*, T_2^* is more efficient than the transformation T_1, T_2, but the maintenance of the localized repair operations R_1, R_2 (as shown in Fig. 3) is more expensive than the return of rejected product flows $\overline{q}_i$ into the manufacture G (as shown in Fig. 2).

Conclusions

1. The received expressions, similarly like in the [1] work, permits the modeling of desired situations of multi-stage continuous control with application of generalized beta-density for stochastic description of defectivity levels and with selection of desired variant for distribution of returning flows with real probabilities of product classification errors.
2. The generalized beta-distribution offers the broader possibilities when providing the mathematical description of defectivity level during the modeling compare to the beta-distribution which we tried to apply earlier [1]; this advantage depends mainly on the variable interval of defectivity level values. In addition, the generalized beta-distribution may be also applied for the description of other random values, since the values of the interval (v_0, v_1) can be selected virtually without any constraints $(-\infty \leq v_0 \leq v_1 \leq \infty)$.
3. The control scheme with localized repair operations of rejected products (Fig. 3) is more efficient than the scheme in Fig. 2, in which the returning flows are returned back to the manufacture, but the maintenance of individual repair workplaces is more expensive from economical point of view.
4. It is obvious, that in order to select the control scheme in a reasonable manner, it is purposeful to create the function of losses, which would estimate the maintenance costs of control and repair operations and the losses due to defectivity levels of accepted flows, when various probabilities of real errors are present. When minimizing this function under the defined constraints it is possible to select the scheme which would have the minimal total losses.

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R. Kalnius, D. Eidukas. Applications of Generalized Beta-distribution in Quality Control Models // Electronics and Electrical Engineering. – Kaunas: Technologija, 2007. – No. 1(73) – P. 5–12.

Mathematical models of the main stochastic characteristics of the continuous multi-stage control of mechatronic products were created, when the generalized beta-distribution is applied in order to describe the defectivity level. The principles of the stochastic distribution selection according to the empiric data and when applying Johnson and Pearson curve families. The two structures of control schemes are analyzed: one when the flows of discarded products are returned back to the manufacture process and other when the discarded products are repaired in the localized repair operations after each control stage. Mathematical models estimate the first and the second type errors of product classification in control and repair operations, and the efficiency of the control is evaluated according to the transformed densities of defectivity level, defectivity level averages and the magnitudes of the returning flows in the required points of control scheme. It is shown, that the control scheme with localized repair operations performs its functions more efficiently, but the practical implementation of such scheme costs more compare to the scheme in which all the returning flows are directed back to the manufacture. Ill. 5, bibl. 8 (in English; summaries in English, Russian and Lithuanian).

Р. Кальнюс, Д. Эйдукас. Применение обобщенного бета распределения в моделях контроля качества // Электроника и электротехника. – Каунас: Технология, 2007. – № 1(73). – С. 5–12.

Представлены математические модели основных вероятностных характеристик многоступенчатого сплошного контроля мехатронных изделий, когда для описания уровня дефектности применяется обобщенное бета распределение. Обсуждаются принципы подбора вероятных распределений по эмпирическим данным на основе семейства кривых Джонсона или Пирсона. Анализируются две структуры схем контроля: первая, когда потоки забракованных изделий возвращаются на производственный процесс, и вторая, когда забракованные изделия ремонтируются непосредственно после каждой ступени

контроля на локализированных ремонтных операциях. Полученные математические модели учитывают влияние вероятностей ошибок первого и второго рода, возникающих при классификации изделий во время контроля и во время ремонта. Эффективность контроля дефектности, средних значений уровня дефектности а также величин возвратных потоков в нужных точках схемы сплошного контроля. Показано, что схема контроля с локальными ремонтными операциями функционирует более эффективно, но также является и более дорогостоящей. Ил. 5, библ. 8 (на английском языке; рефераты на английском, русском и литовском яз.).

R. Kalnius, D. Eidukas. Apibendrinto beta skirstinio taikymas kokybės kontrolės modeliuose // Elektronika ir elektrotechnika. – Kaunas: Technologija, 2007. – Nr. 1(73). – P. 5–12.

Sudaryti mechatroninių gaminių ištisinės daugiapakopės kontrolės pagrindinių tikimybinių charakteristikų matematiniai modeliai, kai defektingumo lygiui aprašyti taikomas apibendrintas beta skirstinys. Aptarti pagrindiniai tikimybių skirstinių parinkimo pagal empirinius duomenis principai, paremti Džonsono ir Pirsono kreivių šeimomis. Analizuojamos dvi kontrolės schemų struktūros: kai išbrokuotų gaminių srautai grąžinami į gamybos procesą ir kai išbrokuoti gaminiai remontuojami lokalizuotose remonto operacijose po kiekvienos kontrolės pakopos. Matematiniai modeliai įvertina gaminių klasifikavimo pirmos ir antros rūšies klaidas kontrolės ir remonto operacijose, o kontrolės efektyvumas vertinamas pagal transformuotus defektingumo lygio tankius, defektingumo lygio vidurkius ir grįžtamųjų srautų dydžius reikiamuose kontrolės schemos taškuose. Parodyta, kad kontrolės schema su lokalizuotomis remonto operacijomis funkcionuoja efektyviau, tačiau praktiškai įgyvendinti tokią schemą kainuoja brangiau nei schemą, kurioje visi grįžtamieji gaminių srautai grąžinami į gamybą. Il. 5, bibl. 8 (anglų kalba; santraukos anglų, rusų ir lietuvių k.).