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Research article

The auxiliary equation method for the construction of deformed solitary solutions to the model of tumor-immune system interaction

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Abstract: The auxiliary equation method for the construction of deformed solitary solutions to the mathematical model of tumor-immune system interaction is presented in this paper. The investigated model does not admit classical solitary solutions. Special techniques based on symbolic computations are used to construct deformed solitary solutions by simultaneously avoiding the inherent additional constraints to the model parameters. It is demonstrated that the introduction of the auxiliary equation does not drop any solutions from the original system. Also, deformed solitary solutions do not depend on the parameters of the auxiliary equations. Analytical and computations experiments are used to demonstrate the efficacy of the proposed method.

Keywords: soliton; auxiliary equation; uncontrolled dynamics; constraint
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1. Introduction

Mathematical modeling plays a crucial role in understanding cancer dynamics, providing a framework to describe tumor growth and its interactions with the host environment. Among these approaches, phenomenological models based on nonlinear ordinary differential equations (ODEs) occupy a special place, as they effectively capture the time evolution of tumor and immune cell populations and their mutual interactions. Compared to more complex spatial or stochastic models, ODE-based formulations offer a balance between biological interpretability and analytical tractability, making them widely applicable in both theoretical studies and data-driven analyses. They can be used to investigate fundamental growth laws, explore mechanisms of tumor–immune interplay, or predict treatment outcomes under different therapeutic strategies.

ODE-based cancer models can be broadly classified into several categories. Single-population tumor growth models describe the overall expansion of tumor mass and are often used to fit empirical growth curves [1]. Multi-compartment tumor models incorporate heterogeneity within the tumor by distinguishing between proliferative, quiescent, and necrotic cell populations [2]. Tumor–immune interaction models examine processes such as immune surveillance, tumor escape mechanisms, and the stimulation of immune responses by therapy [3]. Tumor–angiogenesis models couple tumor growth to vascular development and nutrient availability [4]. Treatment response models incorporate the cytotoxic effects of chemotherapy, radiotherapy, or immunotherapy, often in combination with pharmacokinetic considerations [5]. Finally, adaptive therapy models focus on personalized treatment strategies and the optimization of therapeutic scheduling to delay resistance and improve patient outcomes [6].

Solitary solutions to nonlinear ordinary differential equations provide an important insight into the global dynamics of relevant systems. Usually, solitary solutions represent the separatrix system that describes the basin boundaries of different attractors [7]. The ability to derive analytical solitary solutions to systems of differential equations that represent models of tumor evolution can help to determine the exact boundaries between the subspaces evolving to opposing states of tumor extinction and host death [8].

The specific techniques mentioned above can be broadly categorized as Ansatz methods, where an assumption is made about the functional form of a solution to simplify the problem [9]. This general framework has proven highly effective for finding exact solutions in a variety of nonlinear systems. For instance, different forms of the Ansatz, such as the trial function method, have been applied to the nonlinear Schrödinger equation with time-dependent coefficients [10], while similarity transformations have been used to find analytical matter-wave solutions to generalized Gross-Pitaevskii equations [11]. The auxiliary equation method presented in this paper can be viewed as a specialized Ansatz, tailored to construct deformed solitary solutions in cases where standard solution forms do not apply.

The theory for the construction of solitary solutions to nonlinear differential equations and their systems dates back to 1895 when the paradigmatic Korteweg and de Vries equation describing one-dimensional shallow water waves was discovered [12]. With the growth of computational power, a large variety of methods based on symbolic computations for the construction of solitary solutions have been developed during in recent decades. The enhanced Sardar sub-equation and generalized Riccati equation methods to explore precise solitary wave solutions for the coupled Higgs field equations have been introduced in [13]. An enhanced algebraic method for the construction of solitary wave solutions for new extended Sakovich equations in fluid dynamics is proposed in [14]. The tanh-coth and energy balance methods is used to construct optical solitary wave solutions to the Kuralay system in [15]. The model expansion method for the exploration of novel solitary wave phenomena in the Klein–Gordon equation is proposed in [16]. The modified auxiliary equation method for the construction of solitary wave solutions to the time-fractional thin-film ferroelectric material equation is presented in [17].

The main objective of this paper is completely different from the methods described above. It is already known that there are no solitary solutions to the tumor-immune system interaction model [18]. This paper investigates the existence of a completely different class of solutions (deformed solitary solutions) to the tumor-immune system interaction model. The auxiliary equation is used not for the modification of the governing system of equations but for the simplification of the flow of symbolic computations.

This paper is organized as follows. The model of the uncontrolled dynamics of the tumor-immune system interaction, and the concept of deformed solitary solutions is introduced in Section 2. The motivation of this study is formulated and clarified in Section 3. The auxiliary equation method for the construction of deformed solitary solutions is introduced in Section 4. The flow of transitions and substitutions resulting from the mapping between system, solution, and auxiliary parameters is presented in Sections 5 and 6. The conditions for the existence of solitary solutions to the original system of nonlinear differential equations are derived in Section 7. Computational experiments and simulations illustrating the concept of the auxiliary equation method are given in Section 8. The counter-example showing the limitations of the auxiliary equation method is provided in Section 9. Finally, the concluding remarks are given in the last section.

2. Preliminaries

2.1. The uncontrolled dynamics of the tumor-immune system interaction

The mathematical model describing the interaction of immunotherapy with chemo- and radiotherapy is given in [18]:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = \xi x F(x) - \theta xy - (\alpha + 2\beta r)xu, \\ \frac{\mathrm{d}y}{\mathrm{d}t} = a(1 - bx)yz + \gamma - \delta y - \kappa yu + \nu y v, \\ \frac{\mathrm{d}z}{\mathrm{d}t} = \sigma x + \psi xu - \lambda z, \\ \frac{\mathrm{d}r}{\mathrm{d}t} = -\rho r + u, \end{cases} \tag{2.1}$$

where t is time, x is tumor volume, y is the immuno-competent cell density, z is tumor antigen, and r is the auxiliary variable to model the effects induced by radiation damage. The term $xF(x)$ specifies the tumor growth model [18]:

$$F(x) = 1 - \frac{x^2}{x_\infty}, \tag{2.2}$$

where x_∞ is the constant tumor carrying capacity.

The following system parameters are defined as constants: ξ represents the tumor growth velocity, θ is the tumor-immune system interaction rate, α and β are the tumor dependent radiosensitivity parameters, a is the tumor antigen stimulated proliferation rate, b is the inverse threshold for tumor suppression, γ is the rate of influx from primary organs, δ is the death rate of T-cells, κ is the chemotherapeutic killing parameter, ν is the immune boost, σ is the intrinsic immunogenicity of the tumor, ψ is the radiotherapy induced immunogenicity, λ is the elimination of antigen by the immune system, ρ is the tumor repair rate.

The uncontrolled dynamics of tumor-immune interaction (without the administration of radiation therapy and without the control of a time-varying immune boost) assumes that $u = 0$ and $v = 0$ [18], which decouples the r-dynamics from the rest of the system (2.1). The uncontrolled dynamics of tumor-immune interaction implies that tumor volume and tumor antigen are coupled with the linear relationship [19]:

$$\lambda z = \sigma x. \tag{2.3}$$

Thus, from a mathematical point of view, the uncontrolled dynamics of (2.1) reduces to two coupled ordinary nonlinear differential equations:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \xi x\left(1 - \frac{x^2}{x_\infty}\right) - \theta xy, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = \frac{a\sigma}{\lambda}\left(1 - bx\right) xy + \gamma - \delta y. \end{cases} \tag{2.4}$$

System (2.1) can be represented in the general form [20]:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = a_1 x + a_3 x^3 + \lambda_x xy, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = b_0 + b_1 y + \lambda_y yx + \mu_y yx^2, \end{cases} \tag{2.5}$$

where $a_1, a_3, b_1, \lambda_x, \lambda_y, \mu_y \in \mathbb{R}$.

2.2. *Solitary solutions and their orders*

The standard form of a solitary solution reads [21]:

$$\begin{aligned} x(t) &= \sigma_x \frac{\prod_{k=1}^{n}\left(\exp\left(\eta\left(t - t_0\right)\right) - x_k\right)}{\prod_{k=1}^{n}\left(\exp\left(\eta\left(t - t_0\right)\right) - t_k\right)}, \\ y(t) &= \sigma_y \frac{\prod_{k=1}^{n}\left(\exp\left(\eta\left(t - t_0\right)\right) - y_k\right)}{\prod_{k=1}^{n}\left(\exp\left(\eta\left(t - t_0\right)\right) - t_k\right)}, \end{aligned} \tag{2.6}$$

where $n \in \mathbb{N}$ is the order of the solitary solution; σ_x, σ_y, η, t_0, x_k, y_k, $t_k \in \mathbb{R}$.

The first-order solitary solution (at $n = 1$) represents a sigmoidal function that describes the transition from one steady state to another steady state through a monotonous trajectory (the kink solitary solution):

$$\begin{aligned} x(t) &= \sigma_x \frac{\exp\left(\eta\left(t - t_0\right)\right) - x_1}{\exp\left(\eta\left(t - t_0\right)\right) - t_1}, \\ y(t) &= \sigma_y \frac{\exp\left(\eta\left(t - t_0\right)\right) - y_1}{\exp\left(\eta\left(t - t_0\right)\right) - t_1}. \end{aligned} \tag{2.7}$$

2.3. *The deformed first-order solitary solution*

It is demonstrated in [20] that solitary solutions (including first-order solitary solutions) to Eq (2.5) do not exist. This manuscript does not reproduce that formal proof of nonexistence; rather, its primary focus is on constructing a different class of solutions, namely, deformed solitary solutions.

The concept of the deformed first-order solitary solution is introduced in [20]:

$$\begin{aligned} x(t) &= \sigma_x \frac{\varphi(t) - x_1}{\varphi(t) - t_1}, \\ y(t) &= \sigma_y \frac{\varphi(t) - y_1}{\varphi(t) - t_1}, \end{aligned} \tag{2.8}$$

where the function $\varphi(t)$ represents the unknown deformation of the time scale. Also, $\lim_{t\to\infty} x(t) = \sigma_x$ and $\lim_{t\to\infty} y(t) = \sigma_y$.

2.4. The role of the deformation function $\varphi(t)$

It is shown in [20] that the deformed solitary solutions may also not satisfy the original system of differential equations (2.5). Therefore, the system of Eq (2.5) is extended to accommodate a larger class of solutions:

$$\begin{cases} f(t)\,\dfrac{\mathrm{d}x}{\mathrm{d}t} = a_1 x + a_3 x^3 + \lambda_x xy, \\ f(t)\,\dfrac{\mathrm{d}y}{\mathrm{d}t} = b_0 + b_1 y + \lambda_y yx + \mu_y yx^2, \end{cases} \tag{2.9}$$

where $f(t)$ is the unknown function that represents this extension.

The transformed time scale

$$\hat{t} = \varphi(t) \tag{2.10}$$

can be used to modify the extended system Eq (2.9) to the following form:

$$\hat{f}(\hat{t})\,\frac{\mathrm{d}\hat{x}}{\mathrm{d}\hat{t}} = a_1\hat{x} + a_3\hat{x}^3 + \lambda_x\hat{x}\hat{y}, \tag{2.11}$$

$$\hat{f}(\hat{t})\,\frac{\mathrm{d}\hat{y}}{\mathrm{d}\hat{t}} = b_0 + b_1\hat{y} + \lambda_y\hat{x}\hat{y} + \mu_y\hat{x}^2\hat{y}, \tag{2.12}$$

where the functions $\hat{f}(\hat{t})$, $f(t)$, and $\varphi(t)$ are coupled with the following relationship [20]:

$$\frac{d\varphi(t)}{dt} = \frac{\hat{f}(\varphi(t))}{f(t)}. \tag{2.13}$$

3. The motivation of this paper

The deformed first-order solitary solutions (in the form of Eq (2.8)) to the extended system (Eq (2.9)) are derived in [20]. Plugging Eqs (2.8) and (2.9) into Eqs (2.11) and (2.12) yields the expression of $\hat{f}(\hat{t})$ [20]:

$$\hat{f}(\hat{t}) = \frac{A_0 + A_1\hat{t} + A_2\hat{t}^2 + A_3\hat{t}^3}{\hat{t} - t_1} \tag{3.1}$$

where $A_k \in \mathbb{R}$, $k = 0, 1, 2, 3$. The derivation of special conditions that equalize A_2 and A_3 to zero helps simplify the expression of $\hat{f}(\hat{t})$ [20]. This simplification enables not only the explicit derivation of the deformation function $\varphi(t)$, but also the construction of deformed first-order solitary solutions to Eq (2.9) at $f(t) = 1$ [20].

Clearly, the additional conditions used in the simplification of $\hat{f}(\hat{t})$ narrow the class of possible solutions to Eq (2.9). Each additional constraint detaches the mathematical results further away from the biological processes described by the clinically validated model in Eq (2.1). Therefore, it would be interesting to explore whether it could be possible to derive deformed solitary solutions to Eq (2.9) without the introduction of any simplifications.

Unfortunately, direct transitions with the unsimplified expression of $\hat{f}(\hat{t})$ are impossible due to the limitations of computer algebra systems (computations fail even on Intel(R) Xeon(R) W-2245 CPU @ 3.90 GHz 512 GB RAM machine).

The main objective of this paper is to present an alternative scheme based on the technique coined as the auxiliary equation method. The general idea of the method is based on the introduction of the additional differential equation to the original system of equations. Surprisingly (if specific conditions are met), the introduction of the auxiliary differential equation does not narrow the class of possible solutions. In other words, the construction of deformed solitary solutions to a system of nonlinear differential equations is split into several meta-steps, described in the following sections.

4. The auxiliary equation method

The symbolic algebra required for these transformations is extensive. The complete, commented Wolfram Mathematica notebook used to derive the results in this paper is available online at this link.

The auxiliary equation method is based on linking the original system to a supplementary, solvable ordinary differential equation that shares the same time deformation function $\hat{f}(\hat{t})$. Its structure is formally defined as follows.

Definition 4.1 (Auxiliary equation)**.** For a system of nonlinear ordinary differential equations of the form given in Eq (2.9), an auxiliary equation is a first-order differential equation for one of the system's unknown functions that satisfies the following conditions:

(1) It takes the form $\hat{f}(\hat{t})\frac{d\hat{x}}{d\hat{t}} = P(\hat{x})$, where $P(\hat{x})$ is a polynomial in $\hat{x}$.
(2) The degree of the polynomial $P(\hat{x})$ must be equal to the highest degree of nonlinearity present in the original system of equations.
(3) The function $\hat{f}(\hat{t})$ is identical to the time-deformation function in Eq (2.9).

The highest degree of nonlinearity in Eq (2.11) is 3 (from the $\hat{x}^3$ term), and it is also 3 in Eq (2.12) (from the $\hat{x}^2\hat{y}$ term). Therefore, according to the definition, the polynomial on the right-hand side of the auxiliary equation must be of degree 3. Choosing $\hat{x}$ as the variable for the auxiliary equation, we have:

The highest degree of nonlinearity in Eq (2.11) is 3 (the term $\hat{x}^3$). The highest degree of nonlinearity in Eq (2.12) is also 3 (the term $\hat{x}^2\hat{y}$). Therefore, the order of polynomial in the auxiliary equation is set to 3.

In fact, there is no difference whether the auxiliary equation is chosen in $\hat{x}$ or $\hat{y}$. Choosing $\hat{x}$ yields the following form of the auxiliary equation:

$$\hat{f}(\hat{t})\frac{d\hat{x}}{d\hat{t}} = d_0 + d_1\hat{x} + d_2\hat{x}^2 + d_3\hat{x}^3. \tag{4.1}$$

Remark 4.1. The requirement in Definition 4.1 that the auxiliary polynomial's degree must match the highest nonlinearity of the system is a critical and necessary condition. If a polynomial of a lower degree is chosen, the resulting algebraic system becomes over-constrained, and as demonstrated by the counter-example in Section 9, it is not possible to find a non-trivial solution. Conversely, if a polynomial of a higher degree were assumed (e.g., degree $m > 3$), the process of satisfying the polynomial identity in $\hat{t}$ would systematically force the coefficients of all terms of degree greater than 3 to be zero. This is because the highest-order terms in the identity would originate exclusively from the auxiliary equation, with no corresponding terms from the original system to balance them. The method, therefore, naturally reduces any assumed higher-order auxiliary equation to the correct degree.

Plugging the first-order solitary solution (the $\hat{x}$ part) and its derivative:

$$\hat{x} = \sigma_x \frac{\hat{t} - x_1}{\hat{t} - t_1}; \quad \frac{d\hat{x}}{dt} = \sigma_x \frac{x_1 - t_1}{(t_1 - \hat{t})^2} \tag{4.2}$$

into Eq (4.1) yields the following expression of $\hat{f}(\hat{t})$:

$$\hat{f}(\hat{t}) = \frac{C_0 + C_1\hat{t} + C_2\hat{t}^2 + C_3\hat{t}^3}{B_0 + B_1\hat{t}} \tag{4.3}$$

where $C_0 = -d_0t_1^3 - x_1\sigma_x(d_1t_1^2 + x_1\sigma_x(d_2t_1 + d_3x_1\sigma_x))$; $C_1 = 3d_0t_1^2 + \sigma_x(d_1t_1(t_1 + 2x_1) + x_1\sigma_x(2d_2t_1 + d_2x_1 + 3d_3x_1\sigma_x))$; $C_2 = -3d_0t_1 - \sigma_x(d_1(2t_1 + x_1) + \sigma_x(d_2t_1 + 2d_2x_1 + 3d_3x_1\sigma_x))$; $C_3 = d_0 + \sigma_x(d_1 + \sigma_x(d_2 + d_3\sigma_x))$; $B_0 = \sigma_x t_1(t_1 - x_1)$; $B_1 = \sigma_x(x_1 - t_1)$.

Equation (4.3) can be simplified to the form presented in Eq (3.1), where $A_0 = \frac{C_0}{\sigma_x(x_1-t_1)}$; $A_1 = \frac{C_1}{\sigma_x(x_1-t_1)}$; $A_2 = \frac{C_2}{\sigma_x(x_1-t_1)}$; $A_3 = \frac{C_3}{\sigma_x(x_1-t_1)}$.

The process described in the present and preceding sections is depicted in Figure 1.

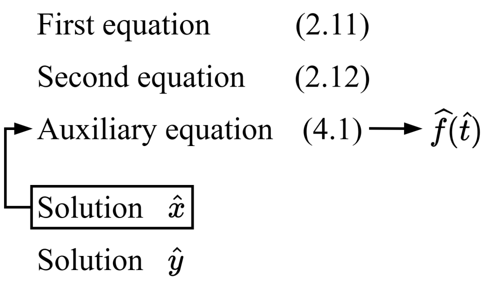


Figure 1. A schematic diagram illustrating the transitions described in Section 4.

5. Coupling the auxiliary equation with Eq (2.11)

Plugging the deformed first-order solitary solution $\hat{x}$ (Eq (4.2)), the expression of $\hat{f}(\hat{t})$ (Eq (3.1)), and the deformed first-order solitary solution $\hat{y}$ and its derivative:

$$\hat{y} = \sigma_y \frac{\hat{t} - y_1}{\hat{t} - t_1}; \quad \frac{d\hat{y}}{dt} = \sigma_y \frac{y_1 - t_1}{(t_1 - \hat{t})^2} \tag{5.1}$$

into Eq (2.11) yields the following identity:

$$\frac{1}{(t_1-\hat{t})^3}\Bigg(d_0(t_1-\hat{t})^3+a_1\sigma_x(t_1-\hat{t})^2(\hat{t}-x_1)+a_3\sigma_x^3(\hat{t}-x_1)^3 \\ -\sigma_x(\hat{t}-x_1)\Big(d_1(t_1-\hat{t})^2+\sigma_x(x_1-\hat{t})(d_2(\hat{t}-t_1)+d_3\sigma_x(\hat{t}-x_1))\Big) \\ -\lambda_x\sigma_x\sigma_y(t_1-\hat{t})(\hat{t}-x_1)(\hat{t}-y_1)\Bigg)=0. \tag{5.2}$$

Collecting the terms at different powers of $\hat{t}$ in the nominator of Eq (5.2) results in the expressions given in Table 1.

Table 1. Terms at the powers of $\hat{t}^k, k=0,1,2,3$ in the numerator of Eq (5.2).

$\hat{t}^0$	$d_0t_1^3+x_1\sigma_x(-a_1t_1^2+d_1t_1^2+d_2t_1x_1\sigma_x-a_3x_1^2\sigma_x^2+d_3x_1^2\sigma_x^2-t_1y_1\lambda_x\sigma_y)$	(a)
$\hat{t}^1$	$-3d_0t_1^2+\sigma_x(a_1t_1(t_1+2x_1)-d_1t_1(t_1+2x_1)-2d_2t_1x_1\sigma_x-d_2x_1^2\sigma_x$ $+3x_1^2\sigma_x^2(a_3-d_3)+t_1x_1\lambda_x\sigma_y+t_1y_1\lambda_x\sigma_y+x_1y_1\lambda_x\sigma_y)$	(b)
$\hat{t}^2$	$3d_0t_1-\sigma_x(a_1(2t_1+x_1)-d_1(2t_1+x_1)-d_2t_1\sigma_x-2d_2x_1\sigma_x$ $+3a_3x_1\sigma_x^2-3d_3x_1\sigma_x^2+t_1\lambda_x\sigma_y+x_1\lambda_x\sigma_y+y_1\lambda_x\sigma_y)$	(c)
$\hat{t}^3$	$-d_0+\sigma_x(a_1-d_1-d_2\sigma_x+a_3\sigma_x^2-d_3\sigma_x^2+\lambda_x\sigma_y)$	(d)

Clearly, Eq (2.11) is satisfied when all expressions in Table 1 are reduced to zero. All other transitions used in this section are based on the step-wise technique for the construction of solitary solutions presented in [22].

5.1. The reduction of the terms at $\hat{t}^3$ (Table 1(d)) to zero.

The terms in Table 1(d) comprise two parameters of the solitary solution σ_x and σ_y (all other parameters are defined by the system of differential equations (2.11), (2.12), and the auxiliary equation (4.1). Clearly, the explicit expression of σ_y is much more simple compared to σ_x:

$$\sigma_y=\frac{d_0+\sigma_x(-a_1+d_1+\sigma_x(d_2-a_3\sigma_x+d_3\sigma_x))}{\lambda_x\sigma_x}. \tag{5.3}$$

Plugging Eq (5.3) into Eq (5.2) and collecting the terms at the powers of $\hat{t}^k$, $k=0,1,2$ in the numerator of Eq (5.2) results in the expressions given in Table 2.

Table 2. Terms at the powers of $\hat{t}^k, k = 0, 1, 2$ in the numerator of Eq (5.2).

$\hat{t}^0$	$d_0t_1^3 + x_1\sigma_x(-a_1t_1^2 + d_1t_1^2 + d_2t_1x_1\sigma_x - a_3x_1^2\sigma_x^2 + d_3x_1^2\sigma_x^2)$ $-t_1x_1y_1(d_0 + \sigma_x(-a_1 + d_1 + \sigma_x(d_2 - a_3\sigma_x + d_3\sigma_x)))$	(a)
$\hat{t}^1$	$-3d_0t_1^2 + a_1t_1\sigma_x((t_1 + 2x_1) - d_1t_1(t_1 + 2x_1) - 2d_2t_1x_1\sigma_x - d_2x_1^2\sigma_x)$ $+3x_1^2\sigma_x^3(a_3 - d_3) + (d_0 + \sigma_x(-a_1 + d_1 + \sigma_x(d_2 - a_3\sigma_x + d_3\sigma_x)))$ $\times(t_1x_1 + t_1y_1 + x_1y_1)$	(b)
$\hat{t}^2$	$d_0(2t_1 - x_1 - y_1) + (t_1 - y_1)(d_1 - a_1)\sigma_x + d_2\sigma_x(x_1 - y_1)$ $+\sigma_x^2(a_3 - d_3)(t_1 - 2x_1 + y_1)$	(c)

5.2. The reduction of the terms at $\hat{t}^2$ (Table 2(c)) to zero.

The free coefficient of the auxiliary equation d_0 is expressed by equalizing the terms in Table 2(c) to zero:

$$d_0 = \frac{(t_1 - y_1)(d_1 - a_1)\sigma_x + d_2\sigma_x(x_1 - y_1) + \sigma_x^2(a_3 - d_3)(t_1 - 2x_1 + y_1)}{x_1 + y_1 - 2t_1}. \quad (5.4)$$

Plugging Eq (5.4) into Eq (5.3) yields the expression of σ_y free from d_0:

$$\sigma_y = \frac{(t_1 - x_1)(a_1 - d_1 + \sigma_x(3\sigma_x(a_3 - d_3) - 2d_2))}{(x_1 + y_1 - 2t_1)\lambda_x}. \quad (5.5)$$

Analogously, plugging Eq (5.4) into Eq (5.2) and collecting the terms at the powers of $\hat{t}^k$, $k = 0, 1$ in the numerator of Eq (5.2) results in the expressions given in Table 3.

Table 3. Terms at the powers of $\hat{t}^k, k = 0, 1$ in the numerator of Eq (5.2).

$\hat{t}^0$	$\frac{(t_1 - x_1)^2\sigma_x}{x_1 + y_1 - 2t_1}((y_1 - t_1)(a_1t_1 - d_1t_1) + d_2t_1\sigma_x(x_1 - y_1)$ $+(a_3 - d_3)(t_1 - x_1)(t_1 + x_1 + y_1)\sigma_x^2)$	(a)
$\hat{t}^1$	$\frac{(t_1 - x_1)^2\sigma_x}{x_1 + y_1 - 2t_1}((y_1 - t_1)(d_1 - a_1) - d_2\sigma_x(x_1 - y_1)$ $-3\sigma_x^2(a_3 - d_3)(t_1 - x_1))$	(b)

5.3. The reduction of the terms at $\hat{t}$ (Table 3(b)) to zero.

The coefficient of the auxiliary equation d_1 is expressed by equalizing the terms in Table 3(b) to zero:

$$d_1 = a_1 + \sigma_x\frac{d_2(y_1 - x_1) + 3\sigma_x(t_1 - x_1)(d_3 - a_3)}{t_1 - y_1}. \quad (5.6)$$

Plugging Eq (5.6) in Eqs (5.4) and (5.5) yields the expressions of d_0 and σ_y free from d_1:

$$d_0 = (a_3 - d_3)\sigma_x^3, \quad (5.7)$$

$$\sigma_y = \frac{\sigma_x(t_1 - x_1)(d_2 + 3\sigma_x(d_3 - a_3))}{\lambda_x(t_1 - y_1)}. \quad (5.8)$$

Analogously, plugging Eqs (5.6)–(5.8) into Eq (5.2) and collecting the terms at $\hat{t}^0$ in the numerator of Eq (5.2) results in the expressions given in Table 4.

Table 4. Terms at $\hat{t}^0$ in the numerator of Eq (5.2).

$\hat{t}^0$	$(a_3 - d_3)(t_1 - x_1)^3\sigma_x^3$	(a)

5.4. The reduction of the terms at $\hat{t}^0$ (Table 4(a)) to zero.

The coefficient of the auxiliary equation d_3 is expressed by equalizing the terms in Table 4(a) to zero:

$$d_3 = a_3. \tag{5.9}$$

Plugging Eq (5.9) in Eqs (5.6)–(5.8) yields the expressions of d_0, d_1, and σ_y free from d_3:

$$d_0 = 0, \tag{5.10}$$

$$d_1 = a_1 - \frac{d_2\sigma_x(x_1 - y_1)}{t_1 - y_1}, \tag{5.11}$$

$$\sigma_y = \frac{d_2\sigma_x(t_1 - x_1)}{\lambda_x(t_1 - y_1)}. \tag{5.12}$$

Clearly, all terms at $\hat{t}^k$, $k = 0, 1, 2, 3$ in the numerator of Eq (5.2) become equal to zero after assuming the condition in Eq (5.9).

Finally, the coefficients A_k, $k = 0, 1, 2, 3$ in Eq (3.1) read:

$$\begin{aligned} A_0 &= \frac{x_1(a_1t_1^2 + \sigma_x(t_1y_1(t_1 - y_1)E + a_3x_1^3\sigma_x))}{t_1 - x_1}, \\ A_1 &= \frac{x_1\sigma_x(d_2(2t_1 + x_1) + 3a_3x_1\sigma_x) + t_1(t_1 + 2x_1)(a_1 - \sigma_x(x_1 - y_1)E)}{x_1 - t_1}, \\ A_2 &= \frac{\sigma_x(d_2(t_1 + 2x_1) + 3a_3x_1\sigma_x) + (2t_1 + x_1)(a_1 - \sigma_x(x_1 - y_1)E)}{t_1 - x_1}, \\ A_3 &= \frac{a_1 + \sigma_x\left(a_3\sigma_x + (t_1 - x_1E)\right)}{x_1 - t_1}, \end{aligned} \tag{5.13}$$

where $E = \frac{d_2}{t_1 - y_1}$.

The process of simplification used in this section is summarized in Figure 2.

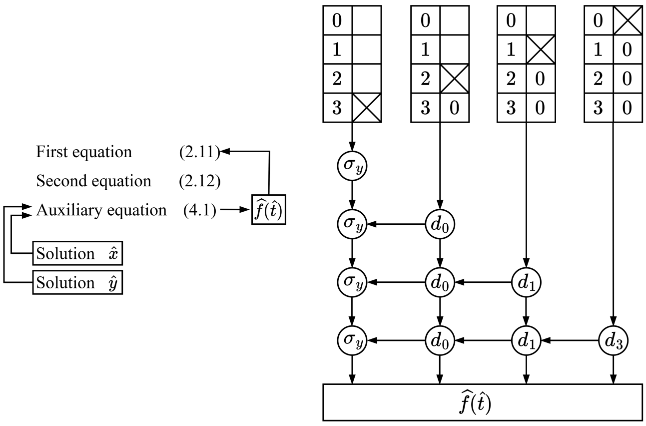


Figure 2. A schematic diagram illustrating the transitions in Section 5.

6. Coupling the auxiliary equation and Eq (2.11) with Eq (2.12)

Let us plug in the $\hat{x}$ and $\hat{y}$ solitary solutions and their derivatives (defined by Eqs (4.2) and (5.1)), the expression of $\hat{f}(\hat{t})$ (defined by Eq (3.1)), and the expressions of coefficients A_k, $k = 0, 1, 2, 3$ (defined by Eq (5.13)) into the second differential equation of the model Eq (2.12). The resulting equality reads:

$$\begin{aligned}
&-b_0 - \frac{d_2\sigma_x}{(t_1-\hat{t})^3(t_1-y_1)\lambda_x}\Bigg(b_1(t_1-\hat{t})^2(t_1-x_1)(y_1-\hat{t}) \\
&+ (t_1-\hat{t})(x_1-\hat{t})(y_1-\hat{t})(t_1-x_1)\lambda_y\sigma_x + (x_1-\hat{t})^2(y_1-\hat{t})(t_1-x_1)\mu_y\sigma_x^2 \\
&+ (x_1-\hat{t})\Big((x_1-\hat{t})(t_1-y_1)\sigma_x\left(d_2(t_1-\hat{t}) + a_3(x_1-\hat{t})\sigma_x\right) \\
&- (t_1-\hat{t})^2(a_1(t_1-y_1) - d_2(x_1-y_1)\sigma_x)\Big)\Bigg) = 0.
\end{aligned} \tag{6.1}$$

Collecting the terms at different powers of $\hat{t}$ in the numerator of Eq (6.1) results in the expressions given in Table 5.

Table 5. Terms at the powers of $\hat{t}^k, k = 0, 1, 2, 3$ in the numerator of Eq (6.1).

$\hat{t}^0$	$-b_0t_1^3(t_1-y_1)\lambda_x + d_2\sigma_x((t_1-y_1)x_1(a_1t_1^2 + a_2x_1^2\sigma_x^2)$ $-(t_1-x_1)y_1(b_1t_1^2 + x_1\sigma_x(t_1\lambda_y - d_2t_1 + x_1\mu_y\sigma_x)))$	(a)
$\hat{t}^1$	$3b_0t_1^2(t_1-y_1)\lambda_x + d_2\sigma_x(-a_1t_1(t_1+2x_1)(t_1-y_1)$ $+b_1t_1(t_1-x_1)(t_1+2y_1) + \sigma_x(-d_2(t_1-x_1)(x_1y_1 + t_1(x_1+y_1))$ $+(t_1-x_1)(x_1y_1 + t_1(x_1+y+1))\lambda_x - x_1(3a_3x_1(t_1-y_1) - (t_1-x_1)(x_1+2y_1)\mu_y)\sigma_x))$	(b)
$\hat{t}^2$	$-3b_0t_1(t_1-y_1)\lambda_x + d_2\sigma_x(a_1(2t_1+x_1)(t_1-y_1) - b_1(t_1-x_1)(2t_1+y_1)$ $+\sigma_x((t_1-x_1)(t_1+x_1+y_1)(d_2-\lambda_y) + 3a_3x_1(t_1-y_1)\sigma_x$ $-(t_1-x_1)(2x_1+y_1)\mu_y\sigma_x))$	(c)
$\hat{t}^3$	$b_0(t_1-y_1)\lambda_x + d_2\sigma_x(b_1(t_1-x_1) - a_1(t_1-y_1)$ $+\sigma_x((t_1-x_1)(\lambda_y-d_2) + ((t_1-x_1)\mu_y - a_3(t_1-y_1))\sigma_x))$	(d)

6.1. The reduction of the terms at $\hat{t}^3$ (Table 5(d)) to zero.

The parameter of the solitary solutions t_1 is expressed by equalizing the terms in Table 5(d) to zero:

$$t_1 = \frac{b_0 y_1 \lambda_x + d_2 x_1 \sigma_x (b_1 - d_2\sigma_x) + d_2\sigma_x(x_1\sigma_x(\lambda_y + \mu_y\sigma_x) - y_1(a_1 + a_3\sigma_x^2))}{b_0\lambda_x + d_2\sigma_x(-a_1 + b_1 + \sigma_x(-d_2 + \lambda_y - \sigma_x(a_3 - \mu_y)))}. \tag{6.2}$$

Plugging Eq (6.2) in Eqs (5.11) and (5.12) yields the expressions of d_1 and σ_y free from t_1:

$$d_1 = \frac{-b_0\lambda_x + d_2\sigma_x(-b_1 + \sigma_x(d_2 - \lambda_y + a_3\sigma_x - \mu_y\sigma_x)) + a_1(b_1 + \sigma_x(\lambda_y + \mu_y\sigma_x))}{b_1 + \sigma_x(d_2 + \lambda_y + \mu_y\sigma_x)}, \tag{6.3}$$

$$\sigma_y = \frac{-b_0\lambda_x + d_2\sigma_x(a_1 + a_3\sigma_x^2)}{\lambda_x(b_1 + \sigma_x(-d_2 + \lambda_y + \mu_y\sigma_x))}. \tag{6.4}$$

6.2. The reduction of the terms at $\hat{t}^2$ to zero.

Plugging the expressions of d_0 (Eq (5.10)), d_1 (Eq (6.3)), d_3 (Eq (5.9)), σ_y (Eq (6.4)), and t_1 (Eq (6.2)) into Eq (6.1) and collecting the terms at $\hat{t}^2$ in the numerator of Eq (6.1) results in the following expression (terms at $\hat{t}^0$ and $\hat{t}^1$ are omitted for brevity):

$$\begin{aligned}&\frac{d_2(x_1 - y_1)^2\sigma_x^2\left(-b_0\lambda_x + d_2\sigma_x(a_1 + a_3\sigma_x^2)\right)}{\left(b_0\lambda_x + d_2\sigma_x(-a_1 + b_1 + \sigma_x(-d_2 + \lambda_y - a_3\sigma_x + \mu_y\sigma_x))\right)^2}\\ &\times\Big(b_1^2 d_2 + b_0 d_2\lambda_x - b_0\lambda_x\lambda_y - 2b_0\lambda_x\mu_y\sigma_x + d_2^3\sigma_x^2 - 2d_2^2\lambda_y\sigma_x^2 + d_2\lambda_y^2\sigma_x^2 + 2a_3 d_2^2\sigma_x^3 - 2d_2^2\mu_y\sigma_x^3\\ &-a_3 d_2\mu_y\sigma_x^4 + d_2\mu_y^2\sigma_x^4 + b_1 b_2\sigma_x(-2d_2 + 2\lambda_y - 3a_3\sigma_x + 2\mu_y\sigma_x) + a_1(-b_1 d_2 + d_2\mu_y\sigma_x^2)\Big).\end{aligned} \tag{6.5}$$

Equating Eq (6.5) to zero yields the following equality:

$$\begin{aligned}b_0 = -&\frac{d_2}{\lambda_x\left(d_2 - \lambda_y - 2\mu_y\sigma_x\right)}\Bigg(b_1^2 + b_1\sigma_x\left(2\lambda_y - 2d_2 + 3a_3\sigma_x + 2\mu_y\sigma_x\right) + a_1\left(\mu_y\sigma_x^2 - b_1\right)\\ &+\sigma_x^2\bigg(d_2^2 + \lambda_y^2 + 2\lambda_y\sigma_x\left(\mu_y - a_3\right) + \mu_y\sigma_x^2\left(\mu_y - a_3\right) - 2d_2\left(\lambda_y + \sigma_x\left(\mu_y - a_3\right)\right)\bigg)\Bigg).\end{aligned} \tag{6.6}$$

Plugging Eq (6.6) into Eqs (6.2)–(6.4) yields the expressions of d_1, σ_y, and t_1 free from b_0:

$$d_1 = \frac{b_1 d_2 - a_1(\lambda_y + 2\mu_y\sigma_x) + d_2\sigma_x(2\lambda_y - 2d_2 - 3a_3\sigma_x + 3\mu_y\sigma_x)}{d_2 - \lambda_y - 2\mu_y\sigma_x}, \tag{6.7}$$

$$\sigma_y = \frac{d_2(b_1 - a_1 + \sigma_x(\lambda_y - d_2 - 3a_3\sigma_x + \mu_y\sigma_x))}{\lambda_x(d_2 - \lambda_y - 2\mu_y\sigma_x)}, \tag{6.8}$$

$$t_1 = \frac{y_1(a_1 - b_1) + (x_1 + y_1)(d_2\sigma_x - \lambda_y\sigma_x) + 3a_3 y_1\sigma_x^2 - \mu_y\sigma_x^2(2x_1 + y_1)}{a_1 - b_1 + \sigma_x(2d_2 - 2\lambda_y + 3\sigma_x(a_3 - \mu_y))}. \tag{6.9}$$

6.3. The reduction of the terms at $\hat{t}$ to zero.

Plugging the expressions of b_0 (Eq (6.6)), d_0 (Eq (5.10)), d_1 (Eq (6.7)), d_3 (Eq (5.9)), σ_y (Eq (6.8)), and t_1 (Eq (6.9)) into Eq (6.1) and collecting the terms at $\hat{t}$ in the numerator of Eq (6.1) results in the following expression (terms at $\hat{t}^0$ are omitted for brevity):

$$\frac{-d_2\sigma_x^3(x_1-y_1)^3(a_1-b_1+\sigma_x(d_2-\lambda_y+\sigma_x(3a_3-\mu_y)))^2}{a_1-b_1+\sigma_x(2d_2-2\lambda_y+3\sigma_x(a_3-\mu_y))^3}\Big((d_2-\lambda_y)^2 \\ +3\sigma_x(a_3-\mu_y)(d_2-\lambda_y)+\mu_y(a_1-b_1-3\sigma_x^2(a_3-\mu_y))\Big). \tag{6.10}$$

Equating Eq (6.10) to zero yields the following equality:

$$\lambda_y = d_2+\frac{3}{2}\sigma_x(a_3-\mu_y)+\frac{1}{2}R, \tag{6.11}$$

where $R = \sqrt{4\mu_y(b_1-a_1)+3(3a_3^2-2a_3\mu_y-\mu_y^2)\sigma_x^2}$. For R to be a real number, the radicand must be non-negative, which imposes a necessary constraint for a physically significant solution. Furthermore, this condition is greatly simplified in later derivations.

Plugging Eq (6.11) into Eq (6.6)–(6.9) yields the expressions of b_0, d_1, σ_y, and t_1 free from λ_y:

$$b_0 = \frac{2b_1d_2(\sigma_xR-a_1+b_1)}{\lambda_x(R+\sigma_x(3a_3+\mu_y))}+\frac{d_2(a_3-\mu_y)\sigma_x^3}{\lambda_x}, \tag{6.12}$$

$$d_1 = \frac{2d_2(a_1-b_1)+a_1\sigma_x(3a_3+\mu_y)+R(a_1-2d_2\sigma_x)}{R+\sigma_x(3a_3+\mu_y)}, \tag{6.13}$$

$$\sigma_y = \frac{d_2(2(a_1-b_1)+\sigma_x(\sigma_x(3a_3+\mu_y)-R))}{\lambda_x(\sigma_x(3a_3+\mu_y)+R)}, \tag{6.14}$$

$$t_1 = \frac{1}{a_1-b_1-\sigma_xR}\big(y_1(a_1-b_1)+d_2\sigma_x(x_1+y_1) \\ +\sigma_x^2(3a_3y_1-2x_1\mu_y-y_1\mu_y-0.5\sigma_x(x_1+y_1)(R+2d_2+3\sigma_x(a_3-\mu_y))\big). \tag{6.15}$$

6.4. The reduction of the terms at $\hat{t}^0$ to zero.

Plugging the expressions of b_0 (Eq (6.12)), d_0 (Eq (5.10)), d_1 (Eq (6.7)), d_3 (Eq (5.9)), λ_y (Eq (6.11)), σ_y (Eq (6.14)), and t_1 (Eq (6.15)) into Eq (6.1) and collecting the terms at $\hat{t}^0$ in the numerator of Eq (6.1) results in the following expression (terms at $\hat{t}^k$, $k = 1, 2, 3$ are already equal to zero):

$$-d_2\sigma_x^4(x_1-y_1)^4(a_3-\mu_y)(3\sigma_x(a_1-b_1)(a_3+\mu_y+R)+2\mu_y\sigma_x^3(3a_3+\mu_y))(2(a_1-b_1)+\sigma_x^2(3a_3+\mu_y)-\sigma_xR)^2. \tag{6.16}$$

Equalizing Eq (6.16) to zero yields the following identity:

$$\mu_y = a_3. \tag{6.17}$$

Note that all terms at $\hat{t}^k$, $k = 0, 1, 2, 3$ in Eq (6.1) are now equal to zero.

The process described in this section is shown in Figure 3.

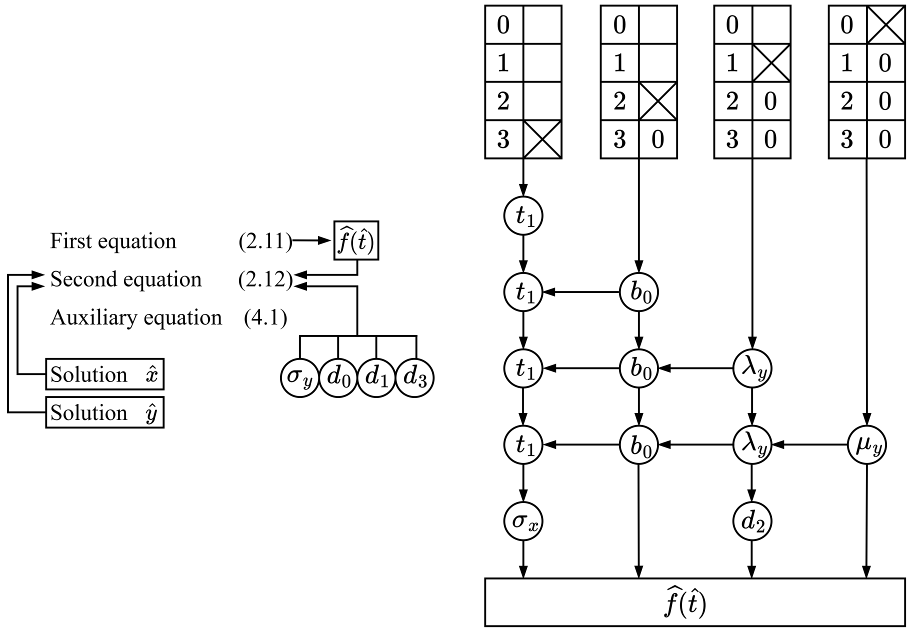


Figure 3. A schematic diagram illustrating the transitions described in Section 6.

7. The existence of solitary solutions to the system of Eqs (2.11) and (2.12)

Plugging Eq (6.17) in Eq (6.11) helps to derive the expression of parameter d_2:

$$d_2 = \lambda_y - \sqrt{a_3(b_1 - a_1)}. \tag{7.1}$$

Plugging the expressions of μ_y (Eq (6.17)) and d_2 (Eq (7.1)) in Eqs (6.12)–(6.15) yields the expressions of b_0, d_1, σ_y, and t_1 free from μ_y and d_2:

$$b_0 = \frac{b_1 d_2 (b_1 - a_1 + 2Q)}{\lambda_x (2a_3\sigma_x + Q)}, \tag{7.2}$$

$$d_1 = \frac{a_1(d_2 + 2a_3\sigma_x + Q) - d_2(b_1 + 2\sigma_x Q)}{2a_3\sigma_x + Q}, \tag{7.3}$$

$$\sigma_y = \frac{d_2(a_1 - b_1 + \sigma_x(2a_3\sigma_x - Q))}{\lambda_x(2a_3\sigma_x + Q)}, \tag{7.4}$$

$$t_1 = \frac{y_1(a_1 - b_1) + 2a_3\sigma_x^2(y_1 - x_1) - \sigma_x Q(x_1 + y_1)}{a_1 - b_1 - 2\sigma_x Q}, \tag{7.5}$$

where $Q = \sqrt{a_3(b_1 - a_1)}$; $a_3 > 0$, $b_1 > a_1$. Note that the expression $a_3(b_1 - a_1)$ is the simplified form of the radicand of R after applying the condition from Eq (6.17). Therefore, the requirement for R to be real simplifies to the constraint $a_3(b_1 - a_1) \geq 0$. For non-trivial solutions, we assume $a_3(b_1 - a_1) > 0$ (e.g., $a_3 > 0$ and $b_1 > a_1$).

The parameter σ_x can be expressed from Eq (7.5):

$$\sigma_x = \frac{Q\sqrt{(2t_1 + x_1 - 3y_1)^2} - Q(x_1 + y_1 - 2t_1)}{4a_3(x_1 - y_1)}. \tag{7.6}$$

Note that Eq (7.5) is a quadratic equation with respect to σ_x, and only one of the two possible solutions is taken in Eq (7.8).

Let us assume that

$$2t_1 + x_1 - 3y_1 > 0. \tag{7.7}$$

Then, Eq (7.6) is reduced to the following form:

$$\sigma_x = \frac{Q(t_1 - y_1)}{a_3(x_1 - y_1)}. \tag{7.8}$$

Then, plugging Eq (7.8) into Eqs (7.2)–(7.4) yields the expressions of b_0, d_1, and σ_y free from σ_x:

$$b_0 = \frac{b_1(a_3(a_1 - b_1) + Q\lambda_y)}{a_3\lambda_x}, \tag{7.9}$$

$$d_1 = \frac{b_1(Q - \lambda_y) + a_1\lambda_y}{Q}, \tag{7.10}$$

$$\sigma_y = \frac{Q(x_1 - t_1)(Q - \lambda_y)}{a_3\lambda_x(x_1 - y_1)}. \tag{7.11}$$

Lemma 7.1. *Introduction of the auxiliary equation (4.1) into the system of Eqs (2.11) and (2.12) does not drop any solutions from the system of Eqs (2.11) and (2.12).*

Proof. The expressions of the coefficients of the auxiliary equation d_0 (Eq (5.10)), d_1 (Eq (7.10)), d_2 (Eq (7.1)), and d_3 (Eq (5.9)) do not depend on the parameters of the solitary solutions x_1, y_1, σ_x, σ_y, and t_1. □

Lemma 7.2. *Solutions to the system of Eqs (2.11) and (2.12) do not depend on the parameters of the auxiliary equation (4.1).*

Proof. The proof follows from the constructive derivation detailed in Sections 5–7. The parameters of the auxiliary equation, $\{d_0, d_1, d_2, d_3\}$, are introduced as free coefficients in Eq (4.1). The systematic process of satisfying the polynomial identity derived from Eq (2.11) (Section 5) explicitly determines the values of d_3, d_0, and d_1 as functions of the system's parameters (a_1, a_3), the solution's parameters, and the remaining free auxiliary parameter d_2 (refer to Eqs (5.9)–(5.11)).

Subsequently, the process of satisfying the polynomial identity from Eq (2.12) (Section 6) imposes further constraints. Crucially, these constraints lead to the final existence conditions on the system parameters (Eqs (6.17) and (7.9)) and determine the solitary solution parameters (Eqs (7.8) and (7.11)) in a manner that is entirely independent of the last free auxiliary parameter, d_2. Since all auxiliary parameters are either determined by the system's structure or are shown to have no influence on the final existence conditions, the resulting solitary solutions do not depend on the specific choice of coefficients in the auxiliary equation, provided it meets the structure laid out in Definition 4.1. □

Theorem 7.1. *Solitary solutions to the system of Eqs (2.11) and (2.12) exist when conditions Eqs (6.17) and (7.9) hold true.*

Proof. The proof is constructive. We begin by assuming that a deformed first-order solitary solution of the form given in Eq (2.8) exists for the system defined by Eqs (2.11) and (2.12). The auxiliary equation method, introduced in Section 4, is then employed. As established by Lemma 7.1, this method does not narrow the class of solutions.

First, the solitary solution forms for $\hat{x}(\hat{t})$ and $\hat{y}(\hat{t})$ are substituted into the first system equation, Eq (2.11). This results in a polynomial identity in the variable $\hat{t}$. For this identity to hold for all $\hat{t}$, the coefficients of each power of $\hat{t}$ must be zero. The systematic process of setting these coefficients to zero, as detailed in Section 5, establishes a set of necessary relationships linking the parameters of the auxiliary equation to the parameters of the original system and the solitary solution.

Next, these derived relationships are carried into the second system equation, Eq (2.12). Again, substituting the solitary solution forms yields another polynomial identity in $\hat{t}$. The process of sequentially setting the coefficients of $\hat{t}^3, \hat{t}^2, \hat{t}^1$, and $\hat{t}^0$ to zero imposes necessary constraints on the parameters of the original system itself. The most critical constraints arise when all coefficients are eliminated, which requires that $\mu_y = a_3$ (Eq (6.17)) and that b_0 adheres to the specific relationship given in Eq (7.9).

When both of these conditions on the system parameters are met, all polynomial equations resulting from the substitution are satisfied identically. This ensures that a consistent set of parameters for the solitary solution (such as σ_x in Eq (7.8) and σ_y in Eq (7.11)) can always be found. This completes the constructive proof. □

7.1. The expressions of parameters A_k, $k = 0, 1, 2, 3$

The expression of coefficients in $\hat{f}(\hat{t})$ is important for the further construction of deformed solitary solutions.

$$A_0 = \frac{x_1(t_1(x_1-y_1)^2(a_1y_1+b_1Q) - x_1(a_1-b_1)(y_1-t_1)^2(y_1Q+t_1\lambda_y))}{Q(t_1-x_1)(x_1-y_1)^2}, \tag{7.12}$$

$$\begin{aligned} A_1 = {} & \frac{1}{Q(x_1-t_1)(x_1-y_1)^2}\Big(t_1(t_1+2x_1)(x_1-y_1)^2(b_1(Q-\lambda_y)+a_1\lambda_y) \\ & -x_1(a_1-b_1)(t_1-y_1)(Q(t_1-x_1)(x_1+2y_1)+\lambda_y(x_1-y_1)(2t_1+x_1))\Big), \end{aligned} \tag{7.13}$$

$$A_2 = \frac{1}{t_1-x_1}\left(\frac{(x_1+2t_1)(Qb_1-\lambda_y)}{Q} + \frac{Q(t_1-y_1)}{a_3(x_1-y_1)}\left(\frac{3Qx_1(t_1-y_1)}{x_1-y_1} + t_1(\lambda_y-Q)(t_1+2x_1)\right)\right), \tag{7.14}$$

$$A_3 = \frac{(a_1\lambda_y+b_1(Q-\lambda_y))(x_1-y_1)^2 - (a_1-b_1)(t_1-y_1)(Qt_1-y_1\lambda_y+x_1(Q+\lambda_y))}{Q(x_1-t_1)(x_1-y_1)^2}. \tag{7.15}$$

8. The construction of solitary solutions to Eq (2.9)

The symbolic transitions and mappings between system parameters, solution parameters, and auxiliary parameters are visualized in Figure 4. As stated in Lemma 7.2, auxiliary parameters d_k, $k = 0, 1, 2, 3$ do not play any role in the structure of the deformed solitary solutions. Auxiliary parameters just help to perform all symbolic transitions. On the other hand, parameters A_k, $k = 0, 1, 2, 3$ define the deformation of the time scale through Eq (2.13).

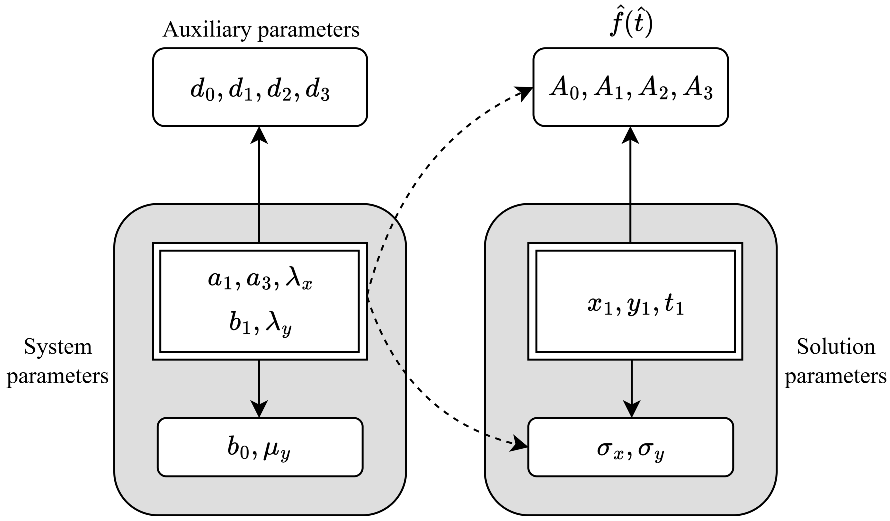


Figure 4. A schematic diagram illustrating the mappings between the system, solution, and auxiliary parameters.

Let us choose the numerical values of the independent system parameters $a_1 = 1$, $a_3 = 2$, $\lambda_x = 5$, $b_1 = 3$, $\lambda_y = 4$, and of the solution parameters $t_1 = -\frac{3}{2}$, $x_1 = -1$, $y_1 = -2$. All chosen numerical values of the parameters to satisfy constraints (6.17) and (7.9). Moreover, the link to tumor-immune data can be reconstructed from Eq (2.4). For example, $a_1 = \xi$ and $a_3 = \frac{\xi}{x_\infty}$.

Then, the dependent system parameters take the following values: $b_0 = \frac{6}{5}$ (according to Eq (7.9)) and $\mu_y = 2$ (according to Eq (6.17)).

The dependent parameters of the solution take the following values: $\sigma_x = \frac{1}{2}$ (according to Eq (7.8)) and $\sigma_y = -\frac{1}{5}$ (according to Eq (7.11)).

The coefficients of $\hat{f}(\hat{t})$ read: $A_0 = -\frac{1}{2}$ (Eq (7.12)), $A_1 = \frac{1}{2}$ (Eq (7.13)), $A_2 = 2$ (Eq (7.14)), $A_3 = 1$ (Eq (7.15)). The parameters of the auxiliary equation read: $d_0 = 0$ (Eq (5.10)), $d_1 = -1$ (Eq (7.10)), $d_2 = 2$ (Eq (7.1)), and $d_3 = 2$ (Eq (5.9)).

The system of differential equations reads:

$$\begin{cases} \hat{f}(\hat{t})\,\dfrac{\mathrm{d}\hat{x}}{\mathrm{d}\hat{t}} = \hat{x} + 2\hat{x}^3 + 5\hat{x}\hat{y}; & (8.1) \\ \hat{f}(\hat{t})\,\dfrac{\mathrm{d}\hat{y}}{\mathrm{d}\hat{t}} = \dfrac{6}{5} + 3\hat{y} + 4\hat{x}\hat{y} + 2\hat{x}^2\hat{y}. & (8.2) \end{cases}$$

The system written above is the transformed system of:

$$\begin{cases} f(t)\,\dfrac{\mathrm{d}x}{\mathrm{d}t} = x + 2x^3 + 5xy; & (8.3) \\ f(t)\,\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{6}{5} + 3y + 4xy + 2x^2y. & (8.4) \end{cases}$$

The auxiliary equation reads:

$$\hat{f}(\hat{t})\frac{d\hat{x}}{d\hat{t}} = -\hat{x} + 2\hat{x}^2 + 2\hat{x}^3, \tag{8.5}$$

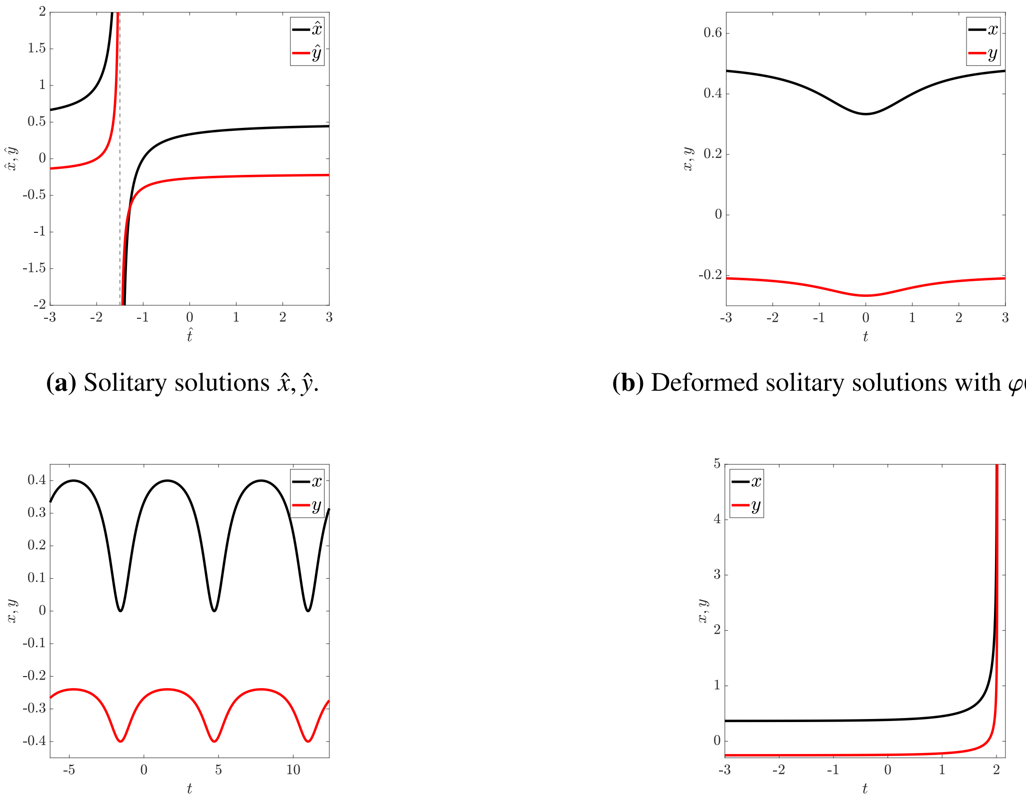
with

$$\hat{f}(\hat{t}) = \frac{(1+\hat{t})(-1+2\hat{t}+2\hat{t}^2)}{3+2\hat{t}}. \tag{8.6}$$

The solitary solutions to Eqs (8.1) and (8.2), $\hat{x}, \hat{y}$ read:

$$\hat{x}\left(\hat{t}\right) = \frac{1+\hat{t}}{3+2\hat{t}}, \quad \hat{y}\left(\hat{t}\right) = -\frac{2}{5}\cdot\frac{2+\hat{t}}{3+2\hat{t}}. \tag{8.7}$$

The above solutions are depicted in Figure 5(a).



(a) Solitary solutions $\hat{x}, \hat{y}$.

(b) Deformed solitary solutions with $\varphi(t) = t^2$.

(c) Deformed solitary solutions with $\varphi(t) = \sin t$.

(d) Deformed solitary solutions with $\varphi(t)$ given implicitly by Eq (8.17).

Figure 5. Plots of deformed and non-deformed solitary solutions obtained in Section 8, Case 1.

8.1. The exp-function transformation $\varphi(t) = e^t$

The nonlinear time transform $\varphi(t) = e^t$ yields $x(t) = \sigma_x \frac{e^t - x_1}{e_t - t_1}$ and $y(t) = \sigma_y \frac{e^t - y_1}{e^t - t_1}$. Then Eq (2.13) yields:

$$f(t) = \frac{e^{-t}}{\sigma_x(t_1 - x_1)(t_1 - e^t)}\Big(d_0(e^t - t_1)^3 + \sigma_x(e^t - x_1)(d_1(e^t - t_1)^2 + \sigma_x(e^t - x_1)(d_2(e^t - t_1) + d_3\sigma_x(e^t - x_1)))\Big). \tag{8.8}$$

More specifically, inserting the parameter values results in:

$$f(t) = \frac{1 - e^{-t} + 4e^t + 2e^{2t}}{3 + 2e^t}. \tag{8.9}$$

Since $f(t) \neq 1$, solitary solutions to Eqs (8.3), (8.4) do not exist. However, as the next subsection demonstrates, other types of solutions can be constructed using $f(t) \neq 1$.

8.2. Transformations $\varphi(t) = t^2, \varphi(t) = \sin t$

Consider the case $\varphi(t) = t^2$. With $\hat{f}\left(\hat{t}\right)$ already given by Eq (8.6), the function $f(t)$ reads:

$$f(t) = \frac{\left(1 + t^2\right)\left(-1 + 2t^2 + 2t^4\right)}{3 + 2t^2}. \tag{8.10}$$

Likewise, the solutions to Eqs (8.3) and (8.4) with the above $f(t)$ are obtained as:

$$x(t) = \frac{1 + t^2}{3 + 2t^2}, \quad y(t) = -\frac{2}{5} \cdot \frac{2 + t^2}{3 + 2t^2}. \tag{8.11}$$

The above functions are depicted in Figure 5(b) and are an example of a deformed solitary solution. This set of solutions also includes solutions with oscillations: They can be obtained by setting $\varphi(t) = \sin t$, yielding:

$$f(t) = \frac{(1 + \sin t)\left(-1 + 2\sin t + 2\sin^2 t\right)}{3 + 2\sin t}, \tag{8.12}$$

and

$$x(t) = \frac{1 + \sin t}{3 + 2\sin t}, \quad y(t) = -\frac{2}{5} \cdot \frac{2 + \sin t}{3 + 2\sin t}. \tag{8.13}$$

The above solutions can be seen in Figure 5(c).

8.3. Deformed solitary solutions with $f(t) = 1$

8.3.1. Case 1

Consider the special case of Eqs (8.3) and (8.4) with $f(t) = 1$:

$$\begin{cases} \dfrac{dx}{dt} = x + 2x^3 + 5xy; & (8.14) \\ \dfrac{dy}{dt} = \dfrac{6}{5} + 3y + 4xy + 2x^2 y. & (8.15) \end{cases}$$

It is clear that this relation has the most applied value, so constructing analytical solutions to this specific case is the most relevant.

Since $\hat{f}\left(\hat{t}\right)$ is given by Eqs (8.5) and (2.13), we have:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{(1+\varphi)\left(-1+2\varphi+2\varphi^2\right)}{3+2\varphi}. \tag{8.16}$$

The above relation cannot be explicitly solved for $\varphi(t)$, but can be rewritten as follows, as long as the integrals involved remain proper:

$$\begin{aligned} &6\ln(\varphi+1) - 3\ln\left(2\varphi^2+2\varphi-1\right) + 2\sqrt{3}\operatorname{arctanh}\left(\frac{\sqrt{3}\,(2\varphi+1)}{3}\right) \\ &- 6\ln\left(\hat{t}_0+1\right) + 3\ln\left(2\hat{t}_0^2+2\hat{t}_0-1\right) - 2\sqrt{3}\operatorname{arctanh}\left(\frac{\sqrt{3}\left(2\hat{t}_0+1\right)}{3}\right) = 6\,(t-t_0)\,. \end{aligned} \tag{8.17}$$

The relation between t_0 and $\hat{t}_0$ follows from the definition of the time transformation:

$$\varphi\,(t_0) = \hat{t}_0. \tag{8.18}$$

Since the original Eqs (8.14) and (8.15) are autonomous, the initial point t_0 can be selected arbitrarily, thus let $t_0 = 0, \hat{t}_0 = 1$.

By numerically evaluating Eq (8.17) to obtain $\varphi(t)$, the solution to Eqs (8.14) and (8.15) can be obtained, as shown in Figure 5(d).

The solution constructed for the case $f(t) = 1$ is of particular importance, as it corresponds to the original uncontrolled dynamics model given in Eq (2.4). This deformed solitary solution represents a special, non-trivial dynamic pathway within the model. The solution connects two distinct asymptotic states: an initial state at $t \to -\infty$ given by $(x, y) = (\sigma_x x_1/t_1, \sigma_y y_1/t_1)$ and a final state at $t \to +\infty$ given by (σ_x, σ_y). In the context of the model, these states can be interpreted as specific population levels of tumor cells (x) and immune cells (y). The analytical form of this solution describes a smooth transition between these two states, offering insight into the potential long-term behaviors of the tumor-immune system under the specific parameter constraints derived in this paper.

A key dynamical feature of the deformed solitary solutions for the case $f(t) = 1$, as plotted in Figure 5(d), is the existence of a finite-time singularity, or blow-up. This behavior is a direct consequence of a simple pole in the solitary solution forms for $\hat{x}(\hat{t})$ and $\hat{y}(\hat{t})$ (see Eq (8.7)), which occurs when the transformed time $\hat{t}$ reaches the parameter value $t_1 = -3/2$. The blow-up in the physical time domain occurs at a finite time t_s when the deformation function reaches this critical value, as determined by solving Eq (8.17) for $\varphi(t_s) = t_1$.

Physically, a blow-up corresponds to infinite tumor volume and immune cell density, which is biologically unrealistic. Therefore, the analytical solution is physically valid and describes the system's dynamics in the time interval leading up to the singularity, $t < t_s$. The singularity itself marks the boundary of the model's applicability, representing a state of uncontrolled growth predicted by the model under the conditions that yield this specific deformed solitary solution.

8.3.2. Case 2

To further validate the presented findings, another special case of the considered system is analyzed:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 4x - x^3 + 5xy, \tag{8.19}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{9}{5} + 3y + 4xy - x^2 y. \tag{8.20}$$

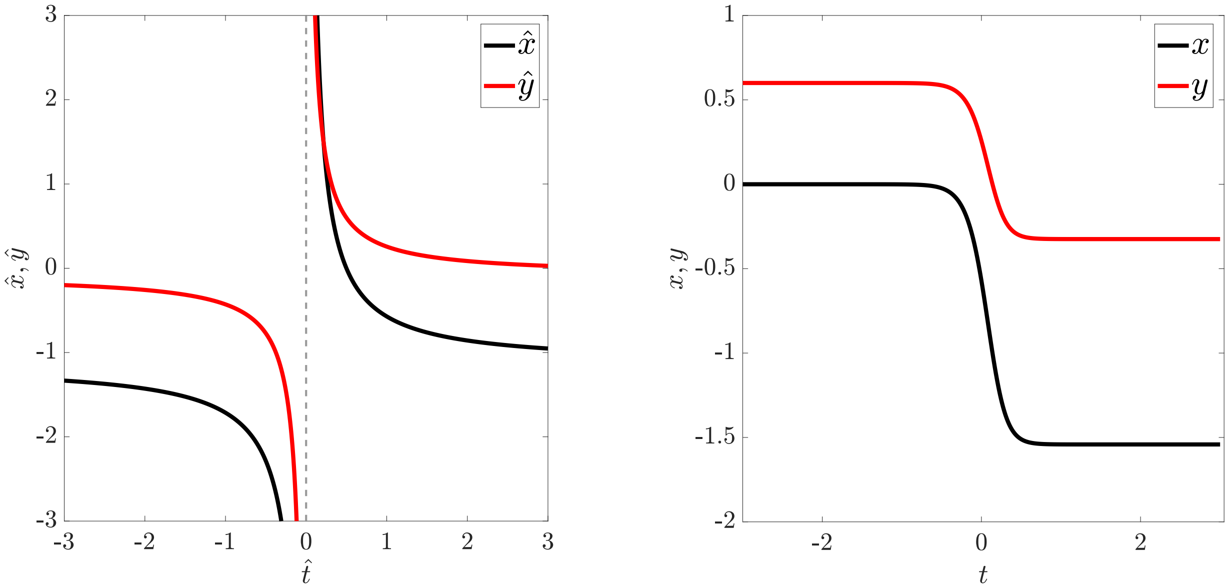
In this case, $\hat{f}$ reads:

$$\hat{f}(\hat{t}) = \frac{\left(-1 - 2\hat{t}\right)\left(-16 + 148\hat{t} + 111\hat{t}^2\right)}{49\hat{t}}, \tag{8.21}$$

while the equation for φ is given by:

$$\begin{aligned} &-\ln(\varphi + 1) + \frac{1}{2}\ln\left(2\varphi^2 + 2\varphi - 1\right) - \frac{\sqrt{3}}{3}\operatorname{arctanh}\left(\frac{\sqrt{3}}{6}(4\varphi + 2)\right) \\ &+ \ln(\varphi_0 + 1) - \frac{1}{2}\ln\left(2\varphi_0^2 + 2\varphi_0 - 1\right) + \frac{\sqrt{3}}{3}\operatorname{arctanh}\left(\frac{\sqrt{3}}{6}(4\varphi_0 + 2)\right) = t - t_0. \end{aligned} \tag{8.22}$$

We will consider $t_0 = 0$ and $\varphi_0 = \varphi(t_0) = 1$. Then, the solutions to (8.19), (8.20) are given in Figure 6. It can be seen that behavior is qualitatively different compared to Case 1: There is no singularity point, and the solutions appear to be similar to non-deformed solitary solutions. However, as the example shows, their analytical expressions are cardinally different.



(a) Solitary solutions $\hat{x}, \hat{y}$.

(b) Deformed solitary solutions with $\varphi(t)$ given by (8.22).

Figure 6. Plots of deformed and non-deformed solitary solutions obtained in Section 8, Case 2.

9. The counter-example

This counter-example demonstrates that a wrong choice of the auxiliary equation fails to produce solitary solutions to the original system of differential equations.

Let us assume that the order of the polynomial of the auxiliary equation is only 2 ($d_3 = 0$):

$$\hat{f}(\hat{t})\frac{d\hat{x}}{d\hat{t}} = d_0 + d_1\hat{x} + d_2\hat{x}^2. \tag{9.1}$$

Plugging Eq (4.2) into Eq (9.1) yields the following expression of $\hat{f}(\hat{t})$:

$$\hat{f}(\hat{t}) = \frac{D_0 + D_1\hat{t} + D_2\hat{t}^2}{E_0}, \tag{9.2}$$

where $D_0 = -d_0t_1^2 - x_1\sigma_x(d_1t_1 + d_2x_1\sigma_x)$, $D_1 = 2d_0t_1 + \sigma_x(d_1(t_1 + x_1) + 2d_2x_1\sigma_x)$, $D_2 = -d_0 - \sigma_x(d_1 + d_2\sigma_x)$, $E_0 = \sigma_x(t_1 - x_1)$.

Plugging $\hat{f}(\hat{t})$ (Eq (9.2)), the $\hat{x}$ solitary solution (Eq (4.2)), and the $\hat{y}$ solitary solution (Eq (5.1)) in the first differential equation (Eq (2.11)) yields an equality analogous to Eq (5.2) ($d_3 = 0$).

Collecting the terms at different powers of $\hat{t}$ in the numerator results in the expressions given in Table 6.

Table 6. Terms at the powers of $\hat{t}^k, k = 0, 1, 2, 3$.

$\hat{t}^0$	$d_0t_1^3 + x_1\sigma_x(-a_1t_1^2 + d_1t_1^2 + d_2t_1x_1\sigma_x - a_3x_1^2\sigma_x^2 - t_1y_1\lambda_x\sigma_y)$	(a)
$\hat{t}^1$	$-3d_0t_1^2 + \sigma_x(a_1t_1(t_1 + 2x_1) - d_1t_1(t_1 + 2x_1) - 2d_2t_1x_1\sigma_x - d_2x_1^2\sigma_x$ $+3x_1^2\sigma_x^2a_3 + t_1x_1\lambda_x\sigma_y + t_1y_1\lambda_x\sigma_y + x_1y_1\lambda_x\sigma_y)$	(b)
$\hat{t}^2$	$3d_0t_1 - \sigma_x(a_1(2t_1 + x_1) - d_1(2t_1 + x_1) - d_2t_1\sigma_x - 2d_2x_1\sigma_x$ $+3a_3x_1\sigma_x^2 + t_1\lambda_x\sigma_y + x_1\lambda_x\sigma_y + y_1\lambda_x\sigma_y)$	(c)
$\hat{t}^3$	$-d_0 + \sigma_x(a_1 - d_1 - d_2\sigma_x + a_3\sigma_x^2 + \lambda_x\sigma_y)$	(d)

Continuing the steps described in Section 5 results in Table 7.

Table 7. Terms at $\hat{t}^0$.

$\hat{t}^0$	$a_3(t_1 - x_1)^3\sigma_x^3$	(a)
$\hat{t}^1$	0	(b)
$\hat{t}^2$	0	(c)
$\hat{t}^3$	0	(d)

It appears that nontrivial solutions to Eq (2.9) do not exist, because setting $a_3 = 0$ would yield a degenerate system of differential equations (2.9), setting $t_1 = x_1$ would yield a trivial solution $\hat{x} = \sigma_x$, and setting $\sigma_x = 0$ would yield a trivial solution $\hat{x} = 0$ (Eq (4.2)).

Failure to comply to the requirements raised for the auxiliary equation prevents the derivation of nontrivial solitary solutions to the original system of differential equations.

10. Discussion on classes of solution construction methods

In general, methods for the construction of solitary solutions to nonlinear differential equations can be classified into analytical, semi-analytical, and numerical methods. The main objective of analytical methods is to derive closed-form exact solitary solutions. Semi-analytical methods aim to construct approximate analytic series (usually infinite) converging to the solitary solution. Numerical methods aim to compute discrete approximations to solitary solutions.

Since the proposed method is analytical, it would be a good idea to compare it to other analytical methods for the construction of solitary solutions which could be conditionally classified into integrability-based methods, direct algebraic methods, symmetry and reduction methods, singularity and expansion methods, variational methods (Table 8).

Special attention must be given to the F-expansion and (G'/G)-expansion methods (belonging to the class of singularity and expansion methods) where the solution to the nonlinear differential equation is expressed as the expansion in terms of known functions satisfying (usually must be simpler) auxiliary equations (elliptic, trigonometric or hyperbolic). The proposed method does not belong to the class of singularity and expansion methods. We do not use any expansions. In contrast, our method belongs to the class of direct algebraic methods. The auxiliary equation is not used to replace or simplify the system. We add the additional differential equation which is balanced up to the highest orders of the nonlinear polynomial terms of the original system.

Moreover, our proposed method goes beyond all the methods mentioned in Table 8. All the methods listed in Table 8 would fail to construct the solitary solution if only solitary solutions do not exist. From this point of view, our method goes beyond and extends the class of direct algebraic methods.

Table 8. A conditional classification of analytical methods for the construction of solitary solutions to nonlinear differential equations.

Class	General idea	Typical method
Integrability-based methods	Applicable for completely integrable PDEs	Lax pair, Darboux transformation, inverse scattering transform
Direct algebraic methods	Travelling wave reduction, solution as a rational function of base functions	Exp-function method, simple equation methos, Hirota bilinear method
Symmetry and reduction methods	Using invariances to reduce the complexity of the system	Lie group and Lie symmetry reductions, scaling transformations
Singularity and expansion methods	Analytic structure of singularities to ensure integrbility	Painleve analysis and truncated expansion, (G'/G)-expansion method
Variational methods	Conservation-law formulations and variational calculus	Energy balance method, variational principle methods

11. Conclusions

The construction of solitary solutions to complex nonlinear models describing the interaction between the tumor and the immune system remains an important mathematical challenge which could also yield some applicable value for the biomedical community. The existence of the system of separatrix shaping the basin boundaries of different attractors could provide an insight into enhanced healing strategies.

We admit that the main contribution of this article is to the mathematical sciences only. The proposed auxiliary equation method helps to tackle difficulties related to the symbolic transformations and transitions. It is important to observe that the proposed method is in a stark contrast to other existing techniques used for the construction of solitary solutions to systems of differential equations describing nonlinear evolutions. From the beginning, we knew that solitary solutions to Eq (2.1) do not exist. Any analytical or computational techniques for the construction of solitary solutions to Eq (2.1) would simply fail.

But we also know that deformed solitary solutions to Eq (2.1) may exist [20]. However, the techniques used in [20] are based on the simplification of the extension function $\hat{f}(\hat{t})$ which automatically induced additional constraints to the parameters of the original system. The main contribution of this paper is the introduction of the new semi-analytical technique (the auxiliary equation method) which enables the relaxation of those constraints. Each additional constraint detaches the mathematical model from the described biological processes. From that point of view, this is the main contribution of this paper to the biological sciences.

While the proposed auxiliary equation technique helps eliminate all the constraints related to the simplification of the extension function $\hat{f}(\hat{t})$, two constraints that relate the parameters of the governing system (2.5) remain in force (Eqs (6.17)) and (7.9)). Thus, although the number of active constraints is significantly reduced (compared to [17]), the conditions for the existence of deformed solitary solutions to (2.5) are not constraint-free.

It is interesting to observe the mathematical implications carried out by these constraints. A system of Riccati equations with multiplicative coupling [23] can be considered as a similar symmetrical counterpart of (2.5):

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = a_0 + a_1 x + a_2 x^2 + a_3 xy, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = b_0 + b_1 y + b_2 y^2 + b_3 xy. \end{cases} \tag{11.1}$$

Solitary solutions (non-deformed) to (11.1) do exist under several constraints, one of which is $a_3 = b_2$ and $b_3 = a_2$ [23]. The constraint $\mu_y = a_3$ can be considered as a non-symmetric reminiscence of these symmetric "cross-like" constraints. In fact, system (2.5) is non-symmetric with respect to nonlinearities. The emerging constraint (6.17) is quite remarkable in that sense.

The biological interpretation of Eq (6.17) is much more complicated. This constraint requires the equality between the normed tumor growth velocity and the inverse threshold of tumor suppression. The biological interpretation of Eq (7.9) is even more complicated. Again, we can only conclude that the major contribution of this paper is the elimination of all constraints related to the structure of the extension function $\hat{f}(\hat{t})$. Biological interpretation of the two remaining constraints remains a definite limitation of this study. However, the proposed approach is a serious achievement from a mathematical point of view.

We also need to admit a number of limitations. First, the scope of this work is focused on the constructive proof of existence for this new class of solutions; a detailed analysis of their uniqueness and stability is not performed but presents a valuable direction for subsequent investigations. All investigations in this paper are focused on the uncontrolled dynamics of tumor immune system interaction (Eq (2.1)). Processes occurring during the treatment intervention (chemotherapy and/or radiotherapy) are ignored in this paper. Another limitation is the considered order of the solitary solution. Classical first-order solitary solutions (kink solitary solutions) are monotonous solutions connecting an initial stationary state with the final stationary state. The connection between the initial state and the final state governed by higher order solitary solutions does not necessarily need to be monotonous. Seeking deformed higher-order solitary solutions to Eq (2.1) remains a definite objective of future research. The specific biological interpretation of deformed solitary solutions (and the connections between deformed solitary solutions and the separatrix system that defines the basin boundaries of different attractors) also remains the objective for future research. That could possibly pave a potential pathway for further development and optimization of treatment models, which is (unfortunately) far from the objectives of this study.

Author contributions

R. Marcinkevičius: software, validation, methodology; R. Mickevičius: literature review, model interpretation; Z. Navickas: conceptualization, formal proofs, methodology; I. Telksnienė: software, validation, writing – review and editing; T. Telksnys: vizualization, methodology, writing – review and editing; M. Ragulskis: supervision, methodology, writing – original draft, writing – review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

Prof. M. Ragulskis is an editorial board member for AIMS Mathematics and was not involved in the editorial review and/or the decision to publish this article.

The authors declare no conflicts of interest.

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