KAUNAS UNIVERSITY OF TECHNOLOGY
VYTAUTAS MAGNUS UNIVERSITY

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# STRUCTURAL DECOMPOSITION OF SECOND ORDER MATRICES IN NONLINEAR SYSTEMS

Summary of doctoral dissertation
Physical sciences, Informatics (09P)

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KAUNO TECHNOLOGIJOS UNIVERSITETAS
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# ANTROS EILĖS MATRICŲ STRUKTŪRINIAI DĖSTINIAI NETIESINIŲ SISTEMŲ TYRIMUOSE

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# INTRODUCTION

## Relevance of the Work

Although the theory of nonlinear dynamical systems derives from work of J.H. Poincaré (19$^{th}$ century), but it is still a subject of great interest and importance. Nonlinear dynamic systems are found in many fields of science: mathematics, physics, biology, economics, and even psychology. Most of the surrounding real systems are complex, and their characterizing signals are multichannel. Examples of such signals are not only biomedical but also seismic, laser techniques or other naturally generated signals. Analysis of individual signals is not sufficient to describe the dynamics of a complex system. Therefore, evaluation of internal signal complexity and mutual signal relations should be investigated.

Standard statistical methods such as correlation, cross-correlation, Granger causality test, mutual information, etc. are commonly used to evaluate relations between two signals. Statistics requires a relatively large amount of data, in addition, it is assumed that the assessed values are random. Naturally, in order to monitor the dynamics of real-time signals, such methods are not suitable.

Since cardiovascular diseases is one of the most common causes of death, and the most widely used non-invasive cardiology test is the electrocardiogram recording, it is crucially important to be able to interpret electrocardiographic changes. A decade ago, a methodology based on matrix analysis was proposed by Lithuanian scientists to evaluate the dynamics of relations between two signals. This methodology was developed and used to investigate the local changes of electrocardiographic signal parameters. The importance of matrix-based methods has been proven by a number of publications and projects. Therefore, a more profound analysis of matrix theory-based characteristics is necessary. This study provides a wider application analysis of previously developed matrix characteristics as well as investigation of new structural matrix parameters. One of the objectives of this research is to provide tools for physicians to evaluate how the complexity of dynamics between two signals is associated with physiological and pathological processes.

Structural matrix analysis-based concept has also been used in iterative maps. The scalar variable was replaced with a second order matrix. An iterative map of matrices shows new dynamical properties, and a modified class of iterative maps was formed. It was noticed that if the matrix of the initial conditions is a nilpotent matrix, then the system may experience divergence; however, if the matrix of the initial conditions is an idempotent matrix, then such an effect is not possible. Iterative maps of matrices may have applications in steganography.

Even though relevance is largely determined by practical applications, the basis for this work is a specific structural decomposition of the second order matrix. Special matrix decomposition-based tools were applied in athletes' physical fitness assessment and clinical research. On the other hand, special decomposition of second order matrices introduced new dynamical effects on the iterative maps of matrices which can be observed neither in scalar iterative maps nor in coupled map lattices. These statements determine the relevance of this dissertation in terms of theoretical value and real-life application capacity.

**The object of the research** is the sequence of second order matrices generated by iterative maps of matrices as well as the sequence obtained from electrocardiographic data.

**The aim of the work** is to investigate special decomposition of second order matrices, to analyze iterative maps of matrices and to observe estimates in order to describe the structure of second order matrices.

**The main tasks of this research are as follows:**

Theoretical

- to develop the scheme to generate the system of idempotents and nilpotents; to introduce parametric expressions of generated idempotents and nilpotents.
- to construct special decomposition of second order matrices and observe the main properties of this decomposition.
- to derive the necessary and sufficient conditions for divergence of the iterative map of matrices.

Application

- to continue and develop the previously investigated (in cooperation with scientists from KTU Biomedical Engineering, LSMU Institute of Cardiology, Institute of Sports and LSU) estimates describing the structure of the second order matrix; to expand and apply the range of estimates so that to analyze the relations between two electrocardiographic parameters.

**Methods, software, and experimental tools:**

- Matrix structural theory is used in the research. MATLAB mathematical and statistical packages were used for comparative analysis.
- Nonlinear dynamical systems models and analysis methods were used to investigate the iterative maps of matrices.
- Structural analysis was applied to investigate the dynamics of relations between two electrocardiographic parameters.

**To be defended:**

- *Special* decomposition of second order matrices based on idempotents and nilpotents.
- *Modified* class of iterative maps of matrices $\mathbf{X}^{(n+1)} = f(\mathbf{X}^{(n)})$ where $\mathbf{X}^{(n)} \in \mathbb{R}_{2\times 2}$ and $f$ is an analytical function. The necessary and sufficient conditions were derived for the divergence of such iterative maps.

- *Novel* estimates based on structural matrix analysis were introduced for the evaluation of the coherence of two signals. These estimates enable to detect electrocardiographic signal changes in comparison with classical ECG analysis methods.

**Scientific novelty and significance**

- *Special* second order square matrix decomposition based on idempotents and nilpotents was introduced during the doctoral research.
- The class of *modified* iterative maps of matrices $\mathbf{X}^{(n+1)} = f(\mathbf{X}^{(n)})$ was constructed, where $\mathbf{X}^{(n)}$ is a second order matrix and $f$ analytic function. Such iterative maps of matrices show effects that cannot be produced by scalar iterative maps or coupled map lattices.
- *Novel* estimates were proposed for the evaluation of relations between two synchronous electrocardiographic signals. The proposed estimates reveal new quality information about signal changes. This methodology is applied in healthcare monitoring and evaluation.

**Approval of the results**

The major results of the dissertation were presented in 12 publications, 5 of which were published in the list of the Institute for Scientific Information (ISI) as the main list of publications with citing indexes. The topics covered in the doctoral dissertation were presented in 7 international conferences. The structural matrix theory-based methodology for ECG analysis was used in five international projects and three patents were taken out for the novel approach in ECG signal processing.

**Scope and structure of the dissertation**

This doctoral dissertation consists of an introduction, 4 major sections, conclusions, practical implications, references and a list of the author's publications. Its total volume is 118 pages. There are 40 figures in the thesis, and the list of cited sources within the main part of the dissertation includes 208 positions.

## 1. LITERATURE REVIEW

*Idempotents and nilpotents as square matrices.* The definitions of idempotent and nilpotent are related to Peirce (1881). Peirce described these elements in algebra: an idempotent is an element which, when multiplied by itself, yields itself and a nilpotent is an element that loses its power and when multiplied by itself yields zero.

The variety of linear combinations of idempotents as square matrices and their properties discussed in the literature emphasizes their high topicality. Properties of rank, trace, eigenvalues and determinants of idempotent and nilpotent combinations are a subject of scholarly discussion. Construction of decomposition of a square matrix is also one of the idempotent and nilpotent-related problems. In this study, one of such problems is covered. The special decomposition of the square matrix of order 2 is constructed. This decomposition is based on the Cayley-Hamilton theorem (Bernstein, 2009).

**Cayley-Hamilton theorem**. If $p(\lambda) = \det(\lambda \mathbf{I} - \mathbf{X})$ is the characteristic polynomial of matrix $\mathbf{X} \in \mathbb{C}_{n \times n}$, then matrix $\mathbf{X}$ also satisfies $p(\mathbf{X}) = \Theta$.

*Chaotic iterative maps*. Many real world systems displaying chaotic behaviour are accurately described with mathematical models. In continuous time, systems are often modelled by differential equations, which is not always convenient for analytical study. Therefore, many studies were devoted to discrete-time dynamical systems (maps).

The *logistic map* is a paradigmatic model often used to demonstrate the onset of chaos and to illustrate how complex behaviour can arise from very simple non-linear dynamical equations (May, 1976):

$$x^{(n+1)} = a x^{(n)} \left(1 - x^{(n)}\right); \tag{1}$$

where $n$ is the iteration number; $n = 0, 1, 2, \ldots$; $a \in \mathbb{R}$ (often investigated $0 \leq a \leq 4$) is the parameter of the logistic map and $x^{(0)}$ is the initial condition (the initial population at year 0). The dynamics of a logistic iterative map depends on parameter $a$. One stable fixed point $x$ when $0 \leq a \leq 1$. One stable fixed point $x = 1 - \frac{1}{a}$ when $1 < a \leq 3$. The first period doubling cascade is when $a = 3$, and fixed point $x = 1 - \frac{1}{a}$ becomes unstable. For $3 < a \leq 3.44949$ $x^{(n)}$ converges to a permanent oscillation between two values that depend on $a$. As $a$ grows larger oscillations between 4 values, then 8, 16, 32, etc. appear. Period-doubling culminates at $a \approx 3.56995$, from where more complex regimes appear.

The *circle map* is a paradigmatic model used to illustrate the effect of phase locking and to study the dynamical behaviour of a beating heart:

$$x^{(n+1)} = x^{(n)} + \Omega - \frac{K}{2\pi} \sin\left(2\pi x^{(n)}\right); \tag{2}$$

where $n$ is the iteration number; model parameter $K$ is the coupling strength and $\Omega$ is the driving phase; initial condition $x^{(0)}$ is a normalized polar angle in the interval $[0,\ 1]$. The circle map exhibits certain regions of parameters $K$ and $\Omega$ where the driving frequency is locked. This phenomenon is called phase locking, and the regions are called Arnold tongues.

A list of chaotic iterative maps is very long. Scalar iterative maps may be connected into coupled map lattices and describe complex high-dimensional dynamics for the entire network. There are also many extensions for scalar iterative maps as well as higher-dimensional iterative maps proposed in scholarly literature.

*Relation between two signals.* Many real world systems are complex, and their produced signals are multivariate. Therefore, inner system relations should be evaluated. Literature offers many methods to evaluate the relationship between simultaneously recorded signals (Pereda et al., 2005).

Correlation function is one of the oldest classical approaches to measure the relation between two variables. Pearson, Spearman's and Kendal correlation coefficients are the most popular. Spearman's rank-order correlation coefficient does not require any special data distribution and assesses monotonic relationships (whether linear or not):

$$r_S = 1 - \frac{6\sum_{i=1}^{n} d_i^2}{n\left(n^2-1\right)}; \tag{3}$$

where $d_i = \text{rank}\left(x_i\right) - \text{rank}\left(y_i\right)$ – is the difference between the two ranks of $i$ observation, and $n$ is the number of observations.

*Granger causality test* is a statistical concept of causality based on prediction. Granger causality test uses autoregressive model in order to evaluate whether prognosis of $X$ becomes better if the information about $Y$ is included in the model:

$$x\left(t\right) = \sum_{k=1}^{p} a_{xyk} x\left(t-k\right) + \sum_{k=1}^{p} b_{xyk} y\left(t-k\right) + \varepsilon_{xy}\left(t\right). \tag{4}$$

If the variance of $\varepsilon_{xy}$ is reduced by the inclusion of the $Y$, then it is said that $Y$ Granger causes $X$ ($Y \to X$). Granger causality $X \to Y$ is described analogously.

Other methods used to evaluate the relation between two signals are cross-correlation functions, nonlinear correlation coefficient, information-theory based methods, and the concept of phase synchronisation.

*Matrix analysis based methodology*. In addition to the existing nonlinear methods, matrix analysis-based methodology was proposed. This methodology was based on phenomenological human as a complex system model proposed by Vainoras (1996). The model implies that periphery (P), regulatory (R) and supplying (S) systems are interconnected (see Fig. 1). Respiratory system (B) may be also added. Relations may be evaluated with certain parameters and measured by using noninvasive procedures: S – systolic arterial blood pressure, D – diastolic systolic arterial blood pressure, RR – time interval between two heartbeats, JT –

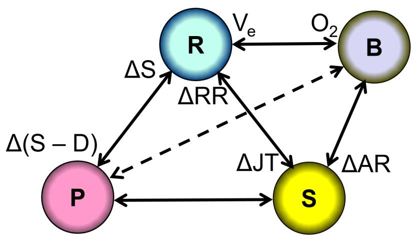


**Figure 1.** Phenomenological human as a complex system model

the time interval from joint point J to the end of T wave. $V_e$ – volume of inhaled air per minute, $O_2$ – volume of used oxygen per minute, AR – R wave amplitude, $\Delta$ means the change of a certain parameter.

As the next step, the methodology based on structural matrix analysis was proposed by Navickas (Vainoras et al., 2008). Let two data sequences be given $y_0, y_1, y_2, \ldots$ and $z_0, z_1, z_2, \ldots$. Then, a sequence of matrices $\mathbf{X}_n$ are constructed:

$$\mathbf{X}_n := \mathbf{X}_n(Y, Z) = \begin{bmatrix} y_n & y_{n-1} - z_{n-1} \\ y_{n+1} - z_{n+1} & z_n \end{bmatrix}; \tag{5}$$

where $n = 1, 2, \ldots$; data sequences $Y$ and $Z$ represent different parameters. Characteristics related to the matrix structure are calculated. One of such characteristics is the matrix discriminant. For each matrix $\mathbf{X}_n$ discriminant is calculated $\operatorname{dsk} \mathbf{X}_n = (y_n - z_n)^2 + 4(y_{n-1} - z_{n-1})(y_{n+1} - z_{n+1})$. The dynamics of $\operatorname{dsk} \mathbf{X}_n$ ($n = 1, 2, \ldots$) is analysed during various physical load tests or procedures in order to evaluate the state of the entire organism, its ability to adapt to the changing environment or to measure body fitness.

## 2. DECOMPOSITION OF SQUARE MATRIX OF ORDER 2

Several properties of square matrices of order 2 will be discussed in this section. These properties are essential before continuing with the decomposition of a matrix.

### 2.1. Basic definitions and notations of matrices

This section contains basic definitions, notations and statements often used in the matrix theory (Bernstein, 2009).Let us consider a square matrix of order 2:

$$\mathbf{X} := \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}, \tag{6}$$

$x_{11}, \ldots, x_{22} \in \mathbb{C}$ and its eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$:

$$\lambda_{1,2} = \frac{1}{2}\left(\operatorname{Tr} \mathbf{X} \pm \sqrt{\operatorname{dsk} \mathbf{X}}\right). \tag{7}$$

where $\operatorname{Tr} \mathbf{X} := x_{11} + x_{22}$; $\operatorname{dsk} \mathbf{X} := (x_{11} - x_{22})^2 + 4x_{12}x_{21}$. If $p_{\mathbf{X}}(z) = 0$ is a characteristic equation as its roots are exactly the eigenvalues of matrix $\mathbf{X}$:

$$\lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1 \cdot \lambda_2 = 0. \tag{8}$$

By the Cayley-Hamilton theorem, $\mathbf{X}$ obeys the same equation

$$\mathbf{X}^2 - (\lambda_1 + \lambda_2)\mathbf{X} + \lambda_1 \cdot \lambda_2 \mathbf{I} = \mathbf{\Theta}; \tag{9}$$

where $\mathbf{\Theta} := \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $\mathbf{I} := \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$.

### 2.2. Idempotents and nilpotents as $2 \times 2$ matrices

Special matrices – idempotents and nilpotents – are presented in this section. Basic properties of idempotents and nilpotents are discussed (Bernstein, 2009; Horn et al., 2012). Proof of properties given in the section can be found in scholarly literature related to matrix analysis.

#### 2.2.1. Properties of idempotents and nilpotents of order 2

**Definition 1.** A square matrix $\mathbf{D}$ is an *idempotent* if satisfies the following:

(i) $$\det \mathbf{D} = 0, \tag{10}$$

(ii) $$\operatorname{Tr} \mathbf{D} = 1. \tag{11}$$

**Property 1.** If matrix $\mathbf{D}$ is an idempotent, then equation $\mathbf{D}^2 = \mathbf{D}$ holds true.

**Remark.** There is a variety of definitions of idempotent. Often, the idempotent is described as a matrix that satisfies equation $\mathbf{D}^2 = \mathbf{D}$. In this study, the definition differs from the classic one by the precondition that the matrices $\mathbf{I}$ and $\mathbf{\Theta}$ cannot be idempotents.

**Property 2.** Idempotents satisfy the following relations:
(i) $\operatorname{Tr} \mathbf{D} = \operatorname{rank} \mathbf{D}$.
(ii) $\mathbf{D}$ eigenvalues are $0$ and $1$,
(iii) If $\mathbf{D}$ is an idempotent, then $\mathbf{I} - \mathbf{D}$ is also an idempotent.

**Definition 2.** Idempotents $\mathbf{D}$ and $\overline{\mathbf{D}}$ that satisfy equation $\mathbf{D} + \overline{\mathbf{D}} = \mathbf{I}$ are *conjugate idempotents*.

**Property 3.** Let the eigenvalues of matrix $\mathbf{X}$ be not equal $\lambda_1 \neq \lambda_2$. Then it is possible to construct two conjugate idempotents $\mathbf{D}_k$:

$$\mathbf{D}_k := \frac{1}{\lambda_k - \lambda_l} \left( \mathbf{X} - \lambda_l \mathbf{I} \right); \tag{12}$$

where $k, l = 1, 2$ and $k \neq l$. Moreover, matrices $\mathbf{D}_k$ satisfy $\mathbf{D}_k \cdot \mathbf{D}_l = \delta_{kl} \mathbf{D}_k$ where $\delta_{kl} := \begin{cases} 1, & k=l; \\ 0, & k \neq l. \end{cases}$

**Definition 3.** Square matrix $\mathbf{N}$ is a *nilpotent* if satisfies the following:

(i) $$\det \mathbf{N} = 0, \tag{13}$$

(ii) $$\operatorname{Tr} \mathbf{N} = 0. \tag{14}$$

**Property 4.** Let matrix $\mathbf{N}$ satisfy (13) and (14) then $\mathbf{N}$ holds true $\mathbf{N}^2 = \Theta$.

**Definition 4.** Nilpotents $\mathbf{N}$ and $c\mathbf{N}$ where $c \in \mathbb{C}$ are *similar nilpotents*.

**Remark.** Matrices $\mathbf{N}$ and $\Theta$ are similar nilpotents.

**Property 5.** Let eigenvalues of matrix $\mathbf{X}$ coincide $\lambda_1 = \lambda_2 = \lambda_0$. Then it is possible to construct nilpotent $\mathbf{N}$:

$$\mathbf{N} := \mathbf{X} - \lambda_0 \mathbf{I}. \tag{15}$$

#### 2.2.2. Sets of $2 \times 2$ idempotents and nilpotents

Matrix $\mathbf{D}$ is an idempotent if it satisfies (10) ir (11). Therefore, it is straightforward to find the whole set of $2 \times 2$ idempotents. Analogously, the set of nilpotents is obtained.

**Property 6.** The set of idempotents can be expressed as:

$$\left\{ \mathbf{D} = \begin{bmatrix} \alpha & \beta \\ \frac{\alpha(1-\alpha)}{\beta} & 1-\alpha \end{bmatrix}, \alpha, \beta \in \mathbb{C}, \beta \neq 0 \right\}. \tag{16}$$

**Remark.** It can be noted that the following conditional limits exist and also provide idempotents that cannot be obtained directly from Eq. (16):

$$\lim_{\substack{\alpha \to 1 \\ \beta \to 0 \\ \left(\frac{\alpha(1-\alpha)}{\beta} \to 0\right)}} \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \quad \lim_{\substack{\alpha \to 0 \\ \beta \to 0 \\ \left(\frac{\alpha(1-\alpha)}{\beta} \to 0\right)}} \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

**Property 7.** The set of nilpotents can be expressed as:

$$\left\{ \mathbf{N} = \begin{bmatrix} \alpha & \beta \\ -\frac{\alpha^2}{\beta} & -\alpha \end{bmatrix}, \alpha, \beta \in \mathbb{C}, \beta \neq 0 \right\}. \tag{17}$$

**Remark.** It can be noted that the following conditional limit provides additional nilpotents to set (17):

$$\lim_{\substack{\alpha \to 0 \\ \beta \to 0 \\ \left(\frac{-\alpha^2}{\beta} \to b\right)}} \mathbf{N} = \lim_{\substack{\alpha \to 0 \\ \beta \to 0 \\ \left(\frac{-\alpha^2}{\beta} \to b\right)}} \begin{bmatrix} \alpha & \beta \\ -\frac{\alpha^2}{\beta} & -\alpha \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix};$$

where $b \in \mathbb{C}$.

### 2.3. Special decomposition of matrix of order 2

This section contains the original scheme to construct the system of two idempotents and two nilpotents. The method is based on splitting a matrix into row matrices and column matrices (Bernstein, 2009). Moreover, the *special* three component decomposition is provided in this section. Special matrix decomposition allows to simplify the calculation of the matrix power.

### 2.3.1. The system of idempotents and nilpotents generated by matrix G

Let $\mathbf{G} := \left[\begin{smallmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{smallmatrix}\right]$ be a nonsingular $\mathbb{C}_{2\times 2}$ matrix. Then an invertible matrix $\mathbf{Y}$ exists:

$$\mathbf{Y} = \mathbf{G}^{-1} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}^{-1} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}. \tag{18}$$

Matrix $\mathbf{G}$ can be divided into column matrices, and matrix $\mathbf{Y}$ can be divided into row matrices (Bernstein, 2009):

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_1 & \mathbf{G}_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} g_{11} \\ g_{21} \end{bmatrix} & \begin{bmatrix} g_{12} \\ g_{22} \end{bmatrix} \end{bmatrix}, \tag{19}$$

$$\mathbf{Y} = \begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} y_{11} & y_{12} \end{bmatrix} \\ \begin{bmatrix} y_{21} & y_{22} \end{bmatrix} \end{bmatrix}. \tag{20}$$

Then matrices $\mathbf{Y}_k$ and $\mathbf{G}_l$ satisfy the following relations

$$\mathbf{Y}_k \mathbf{G}_l = \begin{bmatrix} y_{k1} & y_{k2} \end{bmatrix} \begin{bmatrix} g_{1l} \\ g_{2l} \end{bmatrix} = [\sigma_{kl}] = \begin{cases} 0 & \text{if } k \neq l; \\ 1 & \text{if } k = l; \end{cases} \tag{21}$$

where $k, l = 1, 2$, $\mathbf{Y} = \mathbf{G}^{-1}$ and $\mathbf{Y} \cdot \mathbf{G} = \mathbf{G} \cdot \mathbf{Y} = \mathbf{I}$.

It can be noted that product $\mathbf{G}_k \cdot \mathbf{Y}_l$ yields four different matrices

$$\mathbf{R}_{kl} = \mathbf{G}_k \mathbf{Y}_l = \begin{bmatrix} g_{1k} \\ g_{2k} \end{bmatrix} \begin{bmatrix} y_{l1} & y_{l2} \end{bmatrix}; \tag{22}$$

where $k, l = 1, 2$. Matrices $\mathbf{R}_{11} = \mathbf{D}$ and $\mathbf{R}_{22} = \overline{\mathbf{D}}$ are conjugate idempotents, and matrices $\mathbf{R}_{12} = \mathbf{N}_{12}$ and $\mathbf{R}_{22} = \mathbf{N}_{21}$ are nilpotents. Products of matrices $\mathbf{D}$, $\overline{\mathbf{D}}$, $\mathbf{N}_{12}$, $\mathbf{N}_{21}$, $\mathbf{I}$, $\mathbf{\Theta}$ are given in Table 1.

**Table 1.** Multiplication of matrices $\mathbf{D}, \overline{\mathbf{D}}, \mathbf{N}_{12}, \mathbf{N}_{21}, \mathbf{I}, \mathbf{\Theta}$

| $\cdot$ | $\mathbf{I}$ | $\mathbf{D}$ | $\overline{\mathbf{D}}$ | $\mathbf{N}_{12}$ | $\mathbf{N}_{21}$ | $\Theta$ |
|---|---|---|---|---|---|---|
| $\mathbf{I}$ | $\mathbf{I}$ | $\mathbf{D}$ | $\overline{\mathbf{D}}$ | $\mathbf{N}_{12}$ | $\mathbf{N}_{21}$ | $\Theta$ |
| $\mathbf{D}$ | $\mathbf{D}$ | $\mathbf{D}$ | $\Theta$ | $\mathbf{N}_{12}$ | $\Theta$ | $\Theta$ |
| $\overline{\mathbf{D}}$ | $\overline{\mathbf{D}}$ | $\Theta$ | $\overline{\mathbf{D}}$ | $\Theta$ | $\mathbf{N}_{21}$ | $\Theta$ |
| $\mathbf{N}_{12}$ | $\mathbf{N}_{12}$ | $\Theta$ | $\mathbf{N}_{12}$ | $\Theta$ | $\mathbf{D}$ | $\Theta$ |
| $\mathbf{N}_{21}$ | $\mathbf{N}_{21}$ | $\mathbf{N}_{21}$ | $\Theta$ | $\overline{\mathbf{D}}$ | $\Theta$ | $\Theta$ |
| $\Theta$ | $\Theta$ | $\Theta$ | $\Theta$ | $\Theta$ | $\Theta$ | $\Theta$ |

**Definition 5.** *The system of idempotents and nilpotents generated by matrix* $\mathbf{G}$ is the set of matrices

$$\left\langle \mathbf{G}, \mathbf{G}^{-1} \right\rangle \Rightarrow \left\{ \mathbf{D}, \overline{\mathbf{D}}, \mathbf{N}_{12}, \mathbf{N}_{21} \right\} = S_{\mathbf{G}}. \tag{23}$$

Any matrix $\mathbf{X} := \left[\begin{smallmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{smallmatrix}\right]$ may be expressed as the linear combination of idempotents and nilpotens from system $S_{\mathbf{G}}$

$$\mathbf{X}_{\mathbf{G}} := c_{11}\mathbf{D} + c_{12}\mathbf{N}_{12} + c_{21}\mathbf{N}_{21} + c_{22}\overline{\mathbf{D}}; \tag{24}$$

where $\mathbf{G}$ is a nonsingular matrix.

### 2.3.2. Simple and compound idempotents and their parametric expressions

Two groups of different structure idempotents are discussed in this section.

**Definition 6.** Idempotent $\widetilde{\mathbf{D}}$ is *compound idempotent* if it can be expressed as linear combination $\widetilde{\mathbf{D}} = \mathbf{D} + c\mathbf{N}_{21}$ where $c \in \mathbb{C}$, $\mathbf{D}$ is an idempotent and $\mathbf{N}_{21}$ is a nilpotent from the same system $S_{\mathbf{G}} = \left\{\mathbf{D}, \overline{\mathbf{D}}, \mathbf{N}_{12}, \mathbf{N}_{21}\right\}$.

It can be noted that idempotents (16) are compound idempotents.

**Definition 7.** Idempotents $\mathbf{D}$ and $\overline{\mathbf{D}}$ that can be expressed while using one parameter are *simple idempotents*.

The set of simple idempotents can be obtained from Eq. (16) if one parameter is constant, e.g. $\alpha = \text{const}$. For example, if $\alpha = \frac{1}{2}$ and $2\beta = \beta_0$, then the idempotent is

$$\mathbf{D}^{\langle\beta_0\rangle} = \frac{1}{2}\begin{bmatrix} 1 & \beta_0 \\ \frac{1}{\beta_0} & 1 \end{bmatrix}. \tag{25}$$

A question consequently arises: what is the relations between Eq. (16) and Eq. (25)? It can be noted that linear combination of a simple idempotent and nilpotent covers the whole set of idempotents:

$$\begin{bmatrix} \alpha & \beta \\ \frac{\alpha(1-\alpha)}{\beta} & 1-\alpha \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & \beta_0 \\ \frac{1}{\beta_0} & 1 \end{bmatrix} + c \cdot \frac{1}{2}\begin{bmatrix} 1 & \beta_0 \\ -\frac{1}{\beta_0} & -1 \end{bmatrix}; \tag{26}$$

where $c = 2\alpha - 1$, $\beta_0 = \frac{\beta}{\alpha}$. Equation (26) implies that for every idempotent exists a nilpotent that does not change the rank of the idempotent if added.

### 2.3.3. Three component decomposition of matrix X

Four component decomposition of matrix $\mathbf{X}$ was discussed in the previous section. Then arises the question whether it is possible to reduce the number of components to three. The answer would be positive (Smidtaite et al., 2009; Smidtaite et al., 2010; Navickas et al., 2011). Let $\mathbf{X}$ be a square matrix of order 2, then the three component decomposition of matrix $\mathbf{X}$ is

$$\mathbf{X} := c_{11}\mathbf{D} + c_{21}\mathbf{N}_{21} + c_{22}\overline{\mathbf{D}}; \tag{27}$$

where $\mathbf{D}, \overline{\mathbf{D}}, \mathbf{N}_{21} \in S_{\mathbf{G}}$.

*Matrix three component decomposition.* Let matrix $\mathbf{X} := \left[\begin{smallmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{smallmatrix}\right]$ be given.

**Theorem 1.** Matrix $\mathbf{X}$ can be expressed as

$$\mathbf{X} = c_{11} \cdot \begin{bmatrix} \alpha & \beta \\ \frac{\alpha(1-\alpha)}{\beta} & 1-\alpha \end{bmatrix} + c_{21} \cdot \begin{bmatrix} \alpha & \beta \\ -\frac{\alpha^2}{\beta} & -\alpha \end{bmatrix} + c_{22} \cdot \begin{bmatrix} 1-\alpha & -\beta \\ \frac{\alpha(\alpha-1)}{\beta} & \alpha \end{bmatrix}; \quad (28)$$

where $\alpha \in \mathbb{C} \backslash \{0\}$ and coefficients are calculated

(a) If matrix $\mathbf{X}$ elements are $x_{12} \neq 0$ and $x_{21} \neq 0$, then coefficients take the following form:

$$c_{11} = \frac{1}{2}\left(\operatorname{Tr}\mathbf{X} + \sqrt{\operatorname{dsk}\mathbf{X}}\right) = \lambda_1, \quad (29)$$

$$c_{21} = \left(\frac{x_{11} - x_{22} + \sqrt{\operatorname{dsk}\mathbf{X}}}{2\alpha} - \sqrt{\operatorname{dsk}\mathbf{X}}\right) = \gamma, \quad (30)$$

$$c_{22} = \frac{1}{2}\left(\operatorname{Tr}\mathbf{X} - \sqrt{\operatorname{dsk}\mathbf{X}}\right) = \lambda_2, \quad (31)$$

$$\beta = \frac{-(x_{11} - x_{22}) + \sqrt{\operatorname{dsk}\mathbf{X}}}{2x_{21}} \cdot \alpha = \frac{2x_{12}}{\sqrt{\operatorname{dsk}\mathbf{X}} + x_{11} - x_{22}} \cdot \alpha. \quad (32)$$

(b) If matrix $\mathbf{X}$ elements are $x_{12} = 0$ and $x_{21} \neq 0$, then discriminant is $\operatorname{dsk}\mathbf{X} = (x_{11} - x_{22})^2$. The matrix has two decompositions if $x_{11} \neq x_{22}$:
   1. If $\sqrt{\operatorname{dsk}\mathbf{X}} = -(x_{11} - x_{22})$, then the matrix decomposition has the form of (28) and the coefficients are $c_{11} = x_{22} = \lambda_1$, $c_{22} = x_{11} = \lambda_2$, $c_{21} = -\sqrt{\operatorname{dsk}\mathbf{X}} = x_{11} - x_{22}$, $\beta = \frac{x_{22} - x_{11}}{x_{21}} \cdot \alpha$ where $\alpha \in \mathbb{C} \backslash \{0\}$.
   2. If $\sqrt{\operatorname{dsk}\mathbf{X}} = x_{11} - x_{22}$, then $\beta = 0$ and the limit of (28) (as $\beta \to 0$) is calculated. The coefficients are obtained in a straightforward manner: $c_{11} = x_{11} = \lambda_1$, $c_{22} = x_{22} = \lambda_2$, $c_{21} = \frac{1-\alpha}{\alpha}\sqrt{\operatorname{dsk}\mathbf{X}} = \frac{1-\alpha}{\alpha}(x_{11} - x_{22})$ where $\alpha \in \mathbb{C} \backslash \{0\}$.

(c) If matrix $\mathbf{X}$ elements are $x_{12} \neq 0$ and $x_{21} = 0$, then the discriminant is $\operatorname{dsk}\mathbf{X} = (x_{11} - x_{22})^2$. The matrix has two decompositions if $x_{11} \neq x_{22}$:
   1. If $\sqrt{\operatorname{dsk}\mathbf{X}} = x_{11} - x_{22}$, then $\beta = \frac{x_{12}}{x_{11} - x_{22}} \cdot \alpha$ and the matrix decomposition has the form of (28). The coefficients are calculated in a straightforward manner: $c_{11} = x_{11} = \lambda_1$, $c_{22} = x_{22} = \lambda_2$, $c_{21} = \frac{1-\alpha}{\alpha}(x_{11} - x_{22})$ where $\alpha \in \mathbb{C} \backslash \{0\}$.
   2. If $\sqrt{\operatorname{dsk}\mathbf{X}} = -(x_{11} - x_{22})$, then $\beta \to \infty$ and the limit of (28) is calculated. Then the coefficients $c_{11} = x_{22} = \lambda_1$, $c_{22} = x_{11} = \lambda_2$, $c_{21} = -\sqrt{\operatorname{dsk}\mathbf{X}} = x_{11} - x_{22}$ where $\alpha \in \mathbb{C} \backslash \{0\}$.

(d) If the matrix has the form $\mathbf{X} := \left[\begin{smallmatrix} x_{11} & 0 \\ 0 & x_{22} \end{smallmatrix}\right]$ where $x_{11} \neq x_{22}$, then it has two decompositions:

$$\mathbf{X} = \lim_{\beta \to 0}\left(c_{11}\begin{bmatrix} \alpha & \beta \\ \frac{\alpha(1-\alpha)}{\beta} & 1-\alpha \end{bmatrix} + c_{21}\begin{bmatrix} \alpha & \beta \\ -\frac{\alpha^2}{\beta} & -\alpha \end{bmatrix} + c_{22}\begin{bmatrix} 1-\alpha & -\beta \\ \frac{\alpha(\alpha-1)}{\beta} & \alpha \end{bmatrix}\right); \quad (33)$$

$$\mathbf{X} = \lim_{\beta \to \pm\infty}\left(c_{11}\begin{bmatrix} \alpha & \beta \\ \frac{\alpha(1-\alpha)}{\beta} & 1-\alpha \end{bmatrix} + c_{21}\begin{bmatrix} \alpha & \beta \\ -\frac{\alpha^2}{\beta} & -\alpha \end{bmatrix} + c_{22}\begin{bmatrix} 1-\alpha & -\beta \\ \frac{\alpha(\alpha-1)}{\beta} & \alpha \end{bmatrix}\right); \quad (34)$$

where $\alpha \in \mathbb{C} \setminus \{0\}$ and the coefficients are calculated $c_{11} = \lambda_1 = x_{11}, c_{21} = \left(\frac{x_{11} - x_{22} + \sqrt{\operatorname{dsk}\mathbf{X}}}{2\alpha} - \sqrt{\operatorname{dsk}\mathbf{X}}\right), c_{22} = \lambda_2 = x_{22}, \operatorname{dsk}\mathbf{X} = (x_{11} - x_{22})^2$.

(e) If the matrix has the form $\mathbf{X} := \left[\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right]$ ($x_{11} = x_{22} = x$), then it can be decomposed as:

$$\mathbf{X} = x \cdot \mathbf{D} + x \cdot \overline{\mathbf{D}}; \tag{35}$$

where $\mathbf{D}$ and $\overline{\mathbf{D}}$ are any two conjugate idempotents.

**Remark.** Matrix $\mathbf{X}$ has two decompositions when $\operatorname{dsk}\mathbf{X} \neq 0$ and has one decomposition when $\operatorname{dsk}\mathbf{X} = 0$.

*Powers of matrix X.* Let matrix $\mathbf{X}$ be decomposed into a linear combination of two idempotents and a nilpotent. Then it is highly convenient to use the matrix decomposition for the powers of the matrix.

**Conclusion 1.** Let matrix $\mathbf{X} := \lambda_1 \mathbf{D} + \lambda_2 \overline{\mathbf{D}} + \gamma \mathbf{N}_{21}$ be decomposed. Then the powers of matrix $\mathbf{X}^n$ ($n = 0, 1, 2, \ldots$) can be expressed as:

$$\mathbf{X}^n := \lambda_1^n \mathbf{D} + \lambda_2^n \overline{\mathbf{D}} + \gamma \sum_{i=0}^{n-1} \lambda_1^{n-1-i} \lambda_2^i \mathbf{N}_{21}; \tag{36}$$

where $\mathbf{D}$, $\overline{\mathbf{D}}$ are idempotents and $\mathbf{N}_{21}$ is a nilpotent of matrix $\mathbf{X}$.

*Commuting matrices.* The commutative property of matrix multiplication is important in computing. Therefore, the conditions for two matrices to commute should be noted.

**Conclusion 2.** Two matrices $\mathbf{X}_1$, $\mathbf{X}_2$ commute, $\mathbf{X}_1 \cdot \mathbf{X}_2 = \mathbf{X}_2 \cdot \mathbf{X}_1$, if and only if matrices $\mathbf{X}_1$ and $\mathbf{X}_2$ have the same idempotents and a similar nilpotent.

**Remark.** The scalar matrix commutes with any other matrix.

**Conclusion 3.** Let idempotents of matrices $\mathbf{X}_1$ and $\mathbf{X}_2$ be the same and nilpotents be similar. Then the idempotents of matrices $\mathbf{X}_1 \cdot \mathbf{X}_2$ and $\mathbf{X}_1 + \mathbf{X}_2$ are the same and the nilpotents are similar to the nilpotents of matrices $\mathbf{X}_1$, $\mathbf{X}_2$.

### 2.4. Types of square matrices of order 2

Three component decomposition was discussed in the previous section. Although three component decomposition is universal, yet it is not always convenient. If parameters $\alpha$, $\beta$ are chosen in a particular way, then the three component decomposition is reduced to two component decomposition but some conditions are added. Different parametrization of idempotents and the nilpotent allows to sort all the second order square matrices into three groups: idempotent matrices, nilpotent matrices and scalar matrices. This kind of classification is highly important while investigating the iterative map of matrices (Navickas et al., 2011; Navickas et al., 2012).

### 2.4.1. Idempotent and nilpotent matrices

Let matrix $\mathbf{X} := \left[\begin{smallmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{smallmatrix}\right]$ be given.

**Definition 8.** Matrix $\mathbf{X}$ is *an idempotent matrix* if it can be expressed as:

$$\mathbf{X} = \lambda_1 \mathbf{D}' + \lambda_2 \overline{\mathbf{D}'}; \tag{37}$$

where $\lambda_1$, $\lambda_2$ are eigenvalues of $\mathbf{X}$ and $\mathbf{D}'$, $\overline{\mathbf{D}'}$ are conjugate idempotents of $\mathbf{X}$.

**Definition 9.** Matrix $\mathbf{X}$ is *a nilpotent matrix* if it can be expressed as:

$$\mathbf{X} = \lambda_0 \mathbf{I} + \mathbf{N}; \tag{38}$$

where $\lambda_1 = \lambda_2 = \lambda_0$ and $\mathbf{N}$ is a nilpotent of $\mathbf{X}$.

**Conclusion 4.** If $\lambda_1 \neq \lambda_2$ (dsk $\mathbf{X} \neq 0$) then matrix $\mathbf{X}$ is an idempotent matrix. If $\lambda_1 = \lambda_2 = \lambda_0$ (dsk $\mathbf{X} = 0$) and $\mathbf{X} - \lambda_0 \mathbf{I} \neq \mathbf{\Theta}$ matrix $\mathbf{X}$ is a nilpotent matrix.

**Remark.** Let us notice that a scalar matrix $\mathbf{X} = \lambda_0 \mathbf{I}$ can be expressed in the form $\lambda_0 \mathbf{I} = \lambda_0 \left(\mathbf{D}' + \overline{\mathbf{D}'}\right) = \lambda_0 \mathbf{D}' + \lambda_0 \overline{\mathbf{D}'}$ where $\mathbf{D}'$, $\overline{\mathbf{D}'}$ are conjugate idempotents. Thus $\mathbf{X} = \lambda_0$ can be interpreted as a nilpotent matrix ($\lambda_0 \mathbf{I} = \lambda_0 \mathbf{I} + \mathbf{\Theta}$) or as an idempotent matrix.

**Definition 10.** Equations (37) and (38) are the *idempotent decomposition* and *nilpotent decomposition* of matrix $\mathbf{X}$.

All square second order matrices $\mathbf{X}$ can be sorted into three types (Fig. 2):
Type I: idempotent matrices where dsk $\mathbf{X} \neq 0$;
Type II: scalar matrices where dsk $\mathbf{X} = 0$ and $\mathbf{X} - \lambda_0 \mathbf{I} = \mathbf{\Theta}$;
Type III: nilpotent matrices where dsk $\mathbf{X} = 0$ and $\mathbf{X} - \lambda_0 \mathbf{I} \neq \mathbf{\Theta}$.

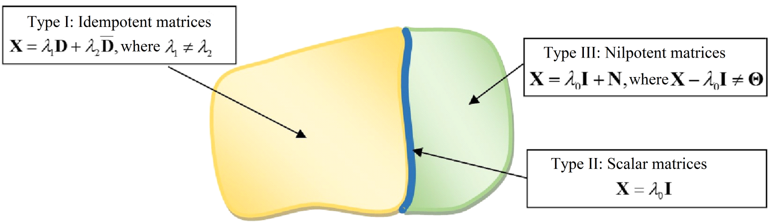


**Figure 2.** The classification of square matrices of order 2

**Conclusion 5.** Let two conjugate idempotents $\mathbf{D}'$, $\overline{\mathbf{D}'}$ and two constants $\lambda_1, \lambda_2 \in \mathbb{C}$ be given. Then $\lambda_1$, $\lambda_2$ are eigenvalues and $\mathbf{D}'$, $\overline{\mathbf{D}'}$ are conjugate idempotents of matrix $\mathbf{X} = \lambda_1 \mathbf{D}' + \lambda_2 \overline{\mathbf{D}'}$.

**Conclusion 6.** Let nilpotent $\mathbf{N}$ and constant $\lambda_0 \in \mathbb{C}$ be given. Then matrix $\mathbf{X} = \lambda_0 \mathbf{I} + \mathbf{N}$ has a single eigenvalue $\lambda_0$ and its nilpotent is $\mathbf{N}$.

**Remark.** Conclusion 2 implies the following:

$$\mathbf{X}_1 \cdot \mathbf{X}_2 = \mathbf{X}_2 \cdot \mathbf{X}_1 = \lambda_{11} \lambda_{21} \mathbf{D}' + \lambda_{12} \lambda_{22} \overline{\mathbf{D}'}; \tag{39}$$

where $\mathbf{X}_1 = \lambda_{11}\mathbf{D}' + \lambda_{12}\overline{\mathbf{D}'}$ and $\mathbf{X}_2 = \lambda_{21}\mathbf{D}' + \lambda_{22}\overline{\mathbf{D}'}$ are idempotent matrices with the same idempotents and

$$\mathbf{X}_1 \cdot \mathbf{X}_2 = \mathbf{X}_2 \cdot \mathbf{X}_1 = \lambda_{10}\lambda_{20}\mathbf{I} + (\lambda_{10}c + \lambda_{20})\,\mathbf{N}; \tag{40}$$

where $\mathbf{X}_1$ and $\mathbf{X}_2$ are nilpotent matrices $\mathbf{X}_1 = \lambda_{10}\mathbf{I} + \mathbf{N}$ and $\mathbf{X}_2 = \lambda_{20}\mathbf{I} + c\mathbf{N}$.

**Remark.** Let us notice that parametric expressions of idempotents and nilpotents $\mathbf{D}'$, $\overline{\mathbf{D}'}$, $\mathbf{N}$ used in (37) and (38) differ from parametric expressions of $\mathbf{D}$, $\overline{\mathbf{D}}$, $\mathbf{N}_{21}$ (27):

$$\mathbf{D}' = \frac{1}{2}\begin{bmatrix} 1+\tilde{\alpha} & \tilde{\beta} \\ \frac{1-\tilde{\alpha}^2}{\tilde{\beta}} & 1-\tilde{\alpha} \end{bmatrix}, \quad \overline{\mathbf{D}'} = \frac{1}{2}\begin{bmatrix} 1-\tilde{\alpha} & -\tilde{\beta} \\ -\frac{1-\tilde{\alpha}^2}{\tilde{\beta}} & 1+\tilde{\alpha} \end{bmatrix}; \tag{41}$$

where $\tilde{\alpha} = 2\alpha - 1$, $\tilde{\beta} = 2\beta$. The nilpotent of nilpotent matrix (38) is

$$\mathbf{N} = \frac{1}{2}\begin{bmatrix} \breve{\alpha} & \breve{\beta} \\ -\frac{\breve{\alpha}^2}{\breve{\beta}} & -\breve{\alpha} \end{bmatrix}; \tag{42}$$

where $\breve{\alpha} = 2\alpha$, $\breve{\beta} = 2\beta$ or simply $\mathbf{N} = \gamma\mathbf{N}_{21}$ where $\gamma = \frac{1}{2}$. The three component decomposition can be drawn to either idempotent decomposition or nilpotent decomposition of matrix $\mathbf{X}$:

1. If $\lambda_1 \neq \lambda_2$ then

$$\mathbf{X} = \lambda_1\mathbf{D} + \lambda_2\overline{\mathbf{D}} + \gamma\mathbf{N}_{21} = \lambda_1\mathbf{D}' + \lambda_2\overline{\mathbf{D}'}; \tag{43}$$

   where $\mathbf{D}' = \mathbf{D} + \frac{\gamma}{\lambda_1-\lambda_2}\mathbf{N}_{21}$, $\overline{\mathbf{D}'} = \overline{\mathbf{D}} - \frac{\gamma}{\lambda_1-\lambda_2}\mathbf{N}_{21}$ are compound idempotents (Section 2.3).

2. If $\lambda_1 = \lambda_2 = \lambda_0$ then

$$\mathbf{X} = \lambda_0\mathbf{D} + \lambda_0\overline{\mathbf{D}} + \gamma\mathbf{N}_{21} = \lambda_0\mathbf{I} + \gamma\mathbf{N}_{21} = \lambda_0\mathbf{I} + \mathbf{N}; \tag{44}$$

   where $\mathbf{N} = \gamma\mathbf{N}_{21}$.

### 2.5. Conclusions

The matrix theory discussed in the second section is oriented to the decomposition of the square matrices of order 2. The importance of the given definitions, properties and theorems is emphasized in the next two sections. The decomposition of second order square matrices is used while investigating iterative maps of matrices as well as analyzing the relations among various parameters of electrocardiogram.

## 3. ITERATIVE MAP OF MATRICES

The iterative map is extended by replacing the scalar variable by a square matrix of variables (Navickas et al., 2011). Dynamical properties of such an iterative map are explored in detail when the order of matrices is 2. It is shown that the evolution of the logistic map depends not only on the control parameters but also on the eigenvalues of the matrix of the initial conditions. Unfortunately, the dynamics of the iterative map with a scalar discrete variable replaced by a square matrix of order 2 already becomes prohibitively complicated. Nevertheless, such variable replacement in the the map introduces specific dynamical effects when the iterative process may diverge at certain eigenvalues of the matrix of the initial conditions (Navickas et al., 2012). Thus, before making any generalizations regarding the dimension of the square matrix of discrete variables, we aim to develop a theory describing the nonlinear dynamics of a general iterative map with a scalar discrete variable replaced by a square matrix of order 2.

### 3.1. Properties of the iterative map of matrices

The iterative map of matrices

$$\mathbf{X}^{(n+1)} := f\left(\mathbf{X}^{(n)}\right); \tag{45}$$

where $n = 0, 1, 2, \ldots$ and matrix $\mathbf{X}^{(n)} = \begin{bmatrix} x_{11}^{(n)} & x_{12}^{(n)} \\ x_{21}^{(n)} & x_{22}^{(n)} \end{bmatrix}$, $x_{kl}^{(n)} \in \mathbb{R}$; $k, l = 1, 2$, function $f : \mathbb{R} \to \mathbb{R}$ is a scalar analytical function.

Let us assume that function $f(x)$ can be expanded into a series:

$$f(x) = \sum_{j=0}^{+\infty} c_j \frac{x^j}{j!}; \tag{46}$$

where $c_j \in \mathbb{R}, j = 0, 1, \ldots$ and $x \in \mathbb{R}$.

**Lemma 1.** Let matrix **X** be an idempotent matrix of order 2 and $f(x)$ can be expressed in the form (46). Then

$$f(\mathbf{X}) := f(\lambda_1)\,\mathbf{D}' + f(\lambda_2)\,\overline{\mathbf{D}'}; \tag{47}$$

where $\lambda_1$, $\lambda_2$ are eigenvalues of **X** $(\lambda_1 \neq \lambda_2)$ and $\mathbf{D}'$, $\overline{\mathbf{D}'}$ are idempotents of **X**.

**Lemma 2.** Let **X** be a nilpotent matrix of order 2 and $f(x)$ can be expressed in the form (46). Then

$$f(\mathbf{X}) = f(\lambda_0)\,\mathbf{I} + f'(\lambda_0)\,\mathbf{N}; \tag{48}$$

where $\lambda_0$ is the recurrent eigenvalue $(\lambda_1 = \lambda_2 = \lambda_0)$ and **N** is the nilpotent of **X**, $f'(\lambda_0)$ denotes the derivative of $f$ with respect to $x$ at $\lambda_0$.

**Conclusion 7.** Let the iterative map read $\mathbf{X}^{(n+1)} := f\left(\mathbf{X}^{(n)}\right) = \sum_{j=0}^{\infty} c_j \frac{\left(\mathbf{X}^{(n)}\right)^j}{j!}$; $n = 0, 1, 2, \ldots$ and the matrix of initial conditions be idempotent matrix $\mathbf{X}^{(0)} = \lambda_1 \mathbf{D}' + \lambda_2 \overline{\mathbf{D}'}$. Then

$$\mathbf{X}^{(n+1)} = \lambda_1^{(n+1)} \mathbf{D}' + \lambda_2^{(n+1)} \overline{\mathbf{D}'} = f\left(\lambda_1^{(n)}\right) \mathbf{D}' + f\left(\lambda_2^{(n)}\right) \overline{\mathbf{D}'}; \tag{49}$$

where $n = 0, 1, 2, \ldots$.

**Remark.** Conclusion 7 yields a straightforward iterative relationship describing the evolution of eigenvalues of the idempotent matrix:

$$\begin{cases} \lambda_1^{(n+1)} = f\left(\lambda_1^{(n)}\right); \\ \lambda_2^{(n+1)} = f\left(\lambda_2^{(n)}\right); \end{cases} \tag{50}$$

where $n = 0, 1, 2, \ldots$. In other words, matrices generated by the iterative map preserve the same idempotents $\mathbf{D}'$, $\overline{\mathbf{D}'}$ if the matrix of initial conditions is an idempotent matrix with idempotents $\mathbf{D}'$ and $\overline{\mathbf{D}'}$.

**Conclusion 8.** Let the iterative map of matrices read $\mathbf{X}^{(n+1)} := f\left(\mathbf{X}^{(n)}\right) = \sum_{j=0}^{\infty} c_j \frac{\left(\mathbf{X}^{(n)}\right)^j}{j!}$; $n = 0, 1, 2, \ldots$ and the matrix is a nilpotent matrix $\mathbf{X}^{(0)} = \lambda_0^{(0)} \mathbf{I} + \mu_0^{(0)} \mathbf{N}$. Then

$$\mathbf{X}^{(n+1)} = \lambda_0^{(n+1)} \mathbf{I} + \mu_0^{(n+1)} \mathbf{N} = f\left(\lambda_0^{(n)}\right) \mathbf{I} + f'\left(\lambda_0^{(n)}\right) \cdot \mu_0^{(n)} \mathbf{N}; \tag{51}$$

where $n = 0, 1, 2, \ldots$ and $\mu_0^{(0)} = 1$.

Let us notice that supplementary variable $\mu_0^{(0)} = 1$ is added in the expression of nilpotent matrix $\mathbf{X}^{(0)}$ in order to keep the same form of matrices of initial conditions $\mathbf{X}^{(0)} = \lambda_0^{(0)} \mathbf{I} + \mu_0^{(0)} \mathbf{N}$ and later $n = 0, 1, 2, \ldots$ iterations (51).

**Remark.** Conclusion 8 yields a straightforward iterative relationship describing the evolution of the eigenvalue of the nilpotent matrix:

$$\begin{cases} \lambda_0^{(n+1)} = f\left(\lambda_0^{(n)}\right); \\ \mu_0^{(n+1)} = f'\left(\lambda_0^{(n)}\right) \cdot \mu_0^{(n)}; \end{cases} \tag{52}$$

where $n = 0, 1, 2, \ldots$ and $\mu_0^{(0)} = 1$. In other words, the iterative map generates a sequence of nilpotent matrices (if only the matrix of initial conditions is a nilpotent matrix). The evolution of the supplementary variable $\mu_0^{(n+1)}$ can be rewritten in the following form:

$$\mu_0^{(n+1)} = \prod_{k=0}^{n} f'\left(\lambda_0^{(k)}\right); \tag{53}$$

where $\mu_0^{(0)} = 1$ and $n = 0, 1, 2, \ldots$.

### 3.2. Logistic map of matrices

The scalar logistic map is well-known and thoroughly explored. A number of extensions of the logistic map have been proposed in academic literature. Therefore, the extension of the logistic iterative map was chosen to investigate first. Discrete scalar variable $x^{(n)}$ in (1) is replaced with a square matrix of order 2; the $n$-th iterate of that matrix is denoted as $\mathbf{X}^{(n)}$ (Navickas et al., 2011; Navickas et al., 2012). Let the matrix of initial conditions read: $\mathbf{X}^{(0)} = \begin{bmatrix} x_{11}^{(0)} & x_{12}^{(0)} \\ :x_{21}^{(0)} & x_{22}^{(0)} \end{bmatrix}$; $x_{kl}^{(0)} \in \mathbb{R}$; $k, l = 1, 2$. Then the iterated map represents a logistic map of square matrices of order 2:

$$\mathbf{X}^{(n+1)} = a\mathbf{X}^{(n)} \left(\mathbf{I} - \mathbf{X}^{(n)}\right) := \begin{bmatrix} x_{11}^{(n+1)} & x_{12}^{(n+1)} \\ x_{21}^{(n+1)} & x_{22}^{(n+1)} \end{bmatrix}. \tag{54}$$

Even though such an extension of the classical logistic map seems to be trivial, the apparent simplicity of the dynamical properties of such an iterative map is misguiding. As an example let us select two different sets of initial conditions and follow the evolution of a four time series (at fixed parameter value $a = 3.7$). Initial conditions $\mathbf{X}^{(0)} = \left[\begin{smallmatrix} 0.2 & 0.3 \\ 0.4 & 0.7 \end{smallmatrix}\right]$ yield 4 fluctuating processes (Fig. 3A – note that some values of the iterated time series are lower than 0). However, initial conditions $\mathbf{X}^{(0)} = \left[\begin{smallmatrix} 0.2 & 0.3 \\ 0.4 & 0.9 \end{smallmatrix}\right]$ yield a violent divergence of iterative processes; numerical overflow is reached after 10 iterations only (Fig. 3B). The primary object of this paper is to explain such dynamic behaviour of the logistic map of matrices when the scalar discrete variable is replaced with a square matrix of order 2.

*Computational experiments*. First of all, it can be noted that the qualitative

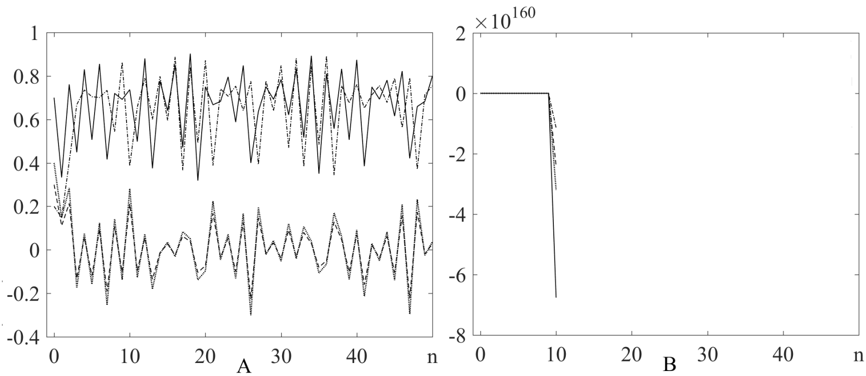


**Figure 3.** The controversy of the logistic map of matrices: $\mathbf{X}^{(0)} = \left[\begin{smallmatrix} 0.2 & 0.3 \\ 0.4 & 0.7 \end{smallmatrix}\right]$ results into 4 stationary processes (A) while $\mathbf{X}^{(0)} = \left[\begin{smallmatrix} 0.2 & 0.3 \\ 0.4 & 0.9 \end{smallmatrix}\right]$ yields a violent divergence of iterative processes (B); $a = 3.7$; -·-·-· stands for $x_{11}^{(n)}$; - - - - stands for $x_{12}^{(n)}$; ·········· stands for $x_{21}^{(n)}$ and —— stands for $x_{22}^{(n)}$

behaviour of iterated matrices of order 2 is governed by Eq. (47) or Eq. (48) (depending on the type of the matrix of initial conditions $\mathbf{X}^{(0)}$). Secondly, it is important to stress that the evolution of the map differs substantially if $\mathbf{X}^{(0)}$ is an idempotent or a nilpotent matrix. If $\mathbf{X}^{(0)}$ has two distinct eigenvalues in $[0,\ 1]$, it is a sufficient condition that the elements of iterated matrices would be bounded for $0 \le a \le 4$. But if $\mathbf{X}^{(0)}$ has one recurrent eigenvalue in $[0,\ 1]$, one can be sure that the elements of iterated matrices would be bounded only for $0 \le a \le 1$; a separate investigation must be done for higher values of parameter $a$.

*Asymptotic versus nonasymptotic convergence;* $1 < a < 3$. Let the matrix of initial conditions be an idempotent matrix and the parameter of logistic map $a$ be bounded in the interval $1 < a < 3$ (a scalar logistic map converges to a stable fixed point $1 - a^{-1}$ then). But then, according to the system of equations (50), both eigenvalues $\lambda_1^{(n)}$ and $\lambda_2^{(n)}$ will converge to $1 - a^{-1}$ at an increasing $n$ (if, of course, $\lambda_1^{(0)}$ and $\lambda_2^{(0)}$ are bounded in the interval $[0,\ 1]$). In other words, the idempotent matrix of the initial conditions will eventually be transformed into a scalar matrix at a sufficiently high $n$. However, such a transformation requires additional explanations which are given below.

First of all, it can be noted that the convergence of a scalar logistic map to a stable fixed point $1 - a^{-1}$ can be asymptotic or nonasymptotic. Let us assume that the current state of the scalar logistic map is $x^{(n)}$. Then a backward iteration from $x^{(n)}$ can be described with the following equality:

$$\left(x^{(n-1)}\right)_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4}{a}x^{(n)}}\right); \tag{55}$$

where the necessary condition for the backward iteration is

$$a - 4x^{(n)} \ge 0. \tag{56}$$

Such a backward iterative process generates a backward tree of points (some branches of the tree are cut as requirement (56) may not always hold). Therefore there exist such points which would yield the exact value of the stable fixed point $1-a^{-1}$ in a finite number of forward iterations (nonasymptotic convergence) (Ragulskis et al., 2011). All the other initial conditions (in the interval $[0,\ 1]$) converge to the fixed point asymptotically.

Figure 4 illustrates asymptotic and nonasymptotic convergence of eigenvalues to a fixed point at $a = 2.5$. The idempotent matrix of initial conditions $\mathbf{X}^{(0)} = \left[\begin{smallmatrix} 0.2 & 0.3 \\ 0.4 & 0.7 \end{smallmatrix}\right]$ is gradually transformed into a scalar matrix: $\lim_{n\to\infty} \mathbf{X}^{(n)} = \left[\begin{smallmatrix} 0.6 & 0 \\ 0 & 0.6 \end{smallmatrix}\right]$ (see Fig. 4A), while its eigenvalues $\lambda_1^{(0)} = 0.023$ and $\lambda_2^{(0)} = 0.877$ converge asymptotically to the fixed point $1 - a^{-1} = 0.6$ (Fig. 4B). Alternatively, the idempotent matrix of initial conditions $\mathbf{X}^{(0)} = \left[\begin{smallmatrix} 2 & -0.6 \\ 3.6 & -1 \end{smallmatrix}\right]$ is transformed into a scalar matrix

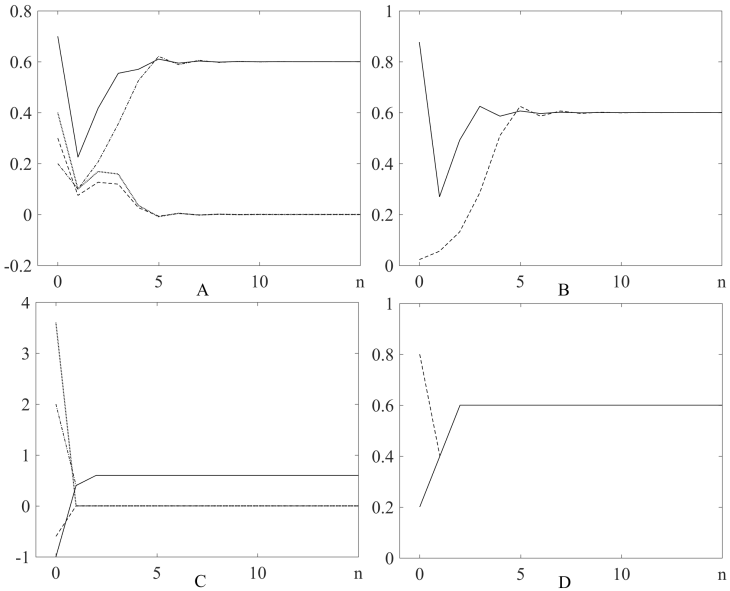


**Figure 4.** Asymptotic versus nonasymptotic convergence to a period-1 attractor: $\mathbf{X}^{(0)} = \left[\begin{smallmatrix} 0.2 & 0.3 \\ 0.4 & 0.7 \end{smallmatrix}\right]$ results into asymptotic convergence (A showing the evolution of $x_{11}^{(n)}$ (-·-·-·-), $x_{12}^{(n)}$ (- - - -), $x_{21}^{(n)}$ (··········), $x_{22}^{(n)}$ (——); B showing the evolution of eigenvalues $\lambda_1^{(n)}$ (- - - -), $\lambda_2^{(n)}$) (- - - -); $\mathbf{X}^{(0)} = \left[\begin{smallmatrix} 2 & -0.6 \\ 3.6 & -1 \end{smallmatrix}\right]$ results into nonasymptotic convergence (C showing the evolution of elements of the matrix and D showing the evolution of its eigenvalues); $a = 2.5$ in both experiments

in two steps: $\mathbf{X}^{(2)} = \left[\begin{smallmatrix} 0.6 & 0 \\ 0 & 0.6 \end{smallmatrix}\right]$ (Fig. 4C) while its eigenvalues $\lambda_1^{(0)} = 0.2$ and $\lambda_2^{(0)} = 0.8$ converge nonasymptotically to 0.6 (Fig. 4D): $\lambda_1^{(1)} = 0.4$; $\lambda_2^{(1)} = 0.4$; $\lambda_1^{(2)} = 0.6$; $\lambda_2^{(2)} = 0.6$.

It can be noted that only two backward iterations were used to construct eigenvalues of $\mathbf{X}^{(0)}$ in this computational example. Of course, more complex examples of nonasymptotic convergence could be used to illustrate the transition from an idempotent matrix to a scalar matrix. In general, if the eigenvalues of matrix $\mathbf{X}^{(n)}$ are $\lambda_1^{(n)}$ and $\lambda_2^{(n)}$ a backward iteration reads:

$$\begin{cases} \left(\lambda_1^{(n-1)}\right)_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4}{a}\lambda_1^{(n)}}\right); \\ \left(\lambda_2^{(n-1)}\right)_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4}{a}\lambda_2^{(n)}}\right). \end{cases} \tag{57}$$

It can be noted that a backward iteration is possible only when $a - 4\lambda_1^{(n)} \geq 0$

and $a - 4\lambda_2^{(n)} \geq 0$. If $\mathbf{X}^{(n)}$ is a nilpotent matrix, a backward iteration reads:

$$\begin{cases} \left(\lambda_0^{(n-1)}\right)_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4}{a}\lambda_0^{(n)}}\right); \\ \left(\mu_0^{(n-1)}\right)_{1,2} = \frac{1}{a\left(1-2\left(\lambda_0^{(n-1)}\right)_{1,2}\right)}\left(\mu_0^{(n)}\right)_{1,2}; \end{cases} \tag{58}$$

and the necessary conditions for a backward iteration are $a - 4\lambda_0^{(n)} \geq 0$ and $0 \leq \left(\lambda_0^{(n-1)}\right)_{1,2} < 1$.

*Periodic attractors at $a = 3.5$; $\mathbf{X}^{(0)}$ is an idempotent matrix.* A period-4 stable attractor exists in a scalar logistic map at $a = 3.5$ (the convergence to this attractor again can be asymptotic or nonasymptotic). Then the following question arises: will any idempotent matrix of the initial conditions evolve into a scalar

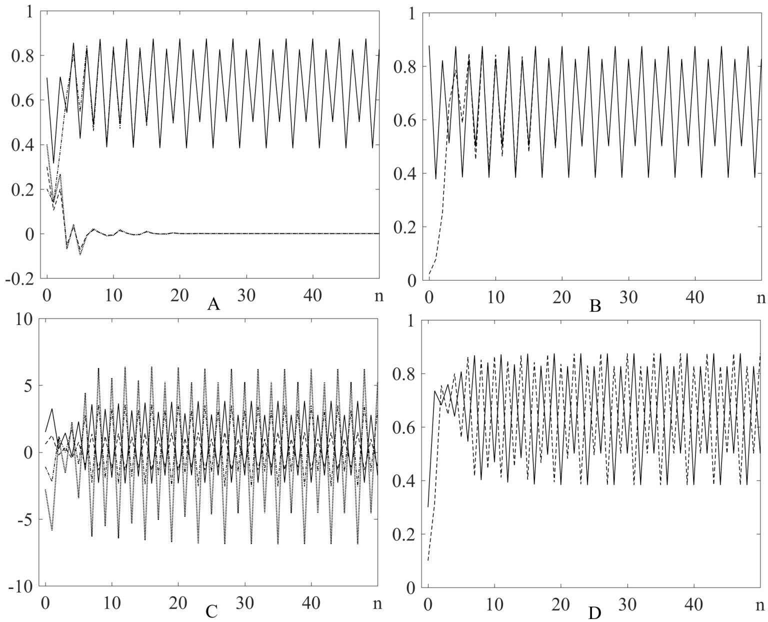


**Figure 5.** An idempotent matrix of the initial conditions can yield a sequence of scalar matrices or a sequence of idempotent matrices: $\mathbf{X}^{(0)} = \left[\begin{smallmatrix} 0.2 & 0.3 \\ 0.4 & 0.7 \end{smallmatrix}\right]$ converges to a sequence of scalar matrices (A showing the evolution of elements of the matrix and B showing the evolution of its eigenvalues) – the phase difference between eigenvalues in the period-4 regime is equal to 0; $\mathbf{X}^{(0)} = \left[\begin{smallmatrix} -1.1 & 0.6 \\ -2.8 & 1.5 \end{smallmatrix}\right]$ yields an infinite sequence of idempotent matrices (C showing the evolution of elements of the matrix and D showing the evolution of its eigenvalues) because eigenvalues converge to the period-4 regime with a phase difference; $a = 3.5$ in both experiments

matrix when eigenvalues will be gradually (or in a finite number of steps) attracted to the period-4 attractor (of course, eigenvalues of $\mathbf{X}^{(0)}$ are bounded in $[0,\ 1]$)?

The answer is negative. Eigenvalues of $\mathbf{X}^{(0)}$ will be attracted to the period-4 attractor in any case, but a phase difference between iterated eigenvalues can be not necessarily equal to zero. This phase difference is constant (and can be equal to 0, 1, 2 or 3 iterates) when both eigenvalues are in the period-4 regime. For example, an idempotent matrix $\mathbf{X}^{(0)} = \left[\begin{smallmatrix} 0.2 & 0.3 \\ 0.4 & 0.7 \end{smallmatrix}\right]$ is gradually transformed into a sequence of scalar matrices (4 different scalar matrices in a period) (Fig. 5A) while its eigenvalues asymptotically converge to the period-4 attractor without a phase difference (Fig. 5B).

However, the idempotent matrix $\mathbf{X}^{(0)} = \left[\begin{smallmatrix} 2 & 3 \\ 4 & 7 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} 0.1 & 0 \\ 0 & 0.3 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} 2 & 3 \\ 4 & 7 \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} -1.1 & 0.6 \\ -2.8 & 1.5 \end{smallmatrix}\right]$ yields an infinite sequence of idempotent matrices because its eigenvalues converge to the period-4 attractor with a constant phase difference not equal to 0 (Fig. 5C and Fig. 5D).

*The evolution of the logistic map of matrices when $X^{(0)}$ is a nilpotent matrix.* A nilpotent matrix of initial conditions $\mathbf{X}^{(0)}$ defined by Eq. (38) will be considered in this section.The values of parameters $\lambda_0^{(0)} = 0.3$; $\breve{\alpha} = 2$ and $\breve{\beta} = 8$ yield $\mathbf{X}^{(0)} = \left[\begin{smallmatrix} 1.3 & 4 \\ -0.25 & -0.7 \end{smallmatrix}\right]$. Fig. 6A and Fig. 6B show strong fluctuations of four scalar

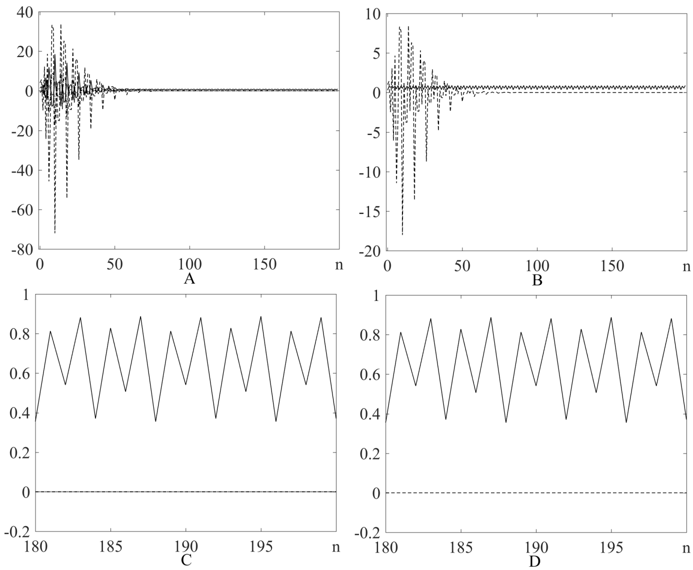


**Figure 6.** The evolution of the logistic map of matrices from $\mathbf{X}^{(0)} = \left[\begin{smallmatrix} 1.3 & 4 \\ -0.25 & -0.7 \end{smallmatrix}\right]$ at $a = 3.5$ (A and C showing the evolution of $x_{11}^{(n)}$, $x_{12}^{(n)}$, $x_{21}^{(n)}$, $x_{22}^{(n)}$; B and D showing the evolution of the eigenvalue (——) and the parameter $\mu_0^{(n)}$) defined by Eq. (53) (----). Evolutions in C and D are displayed in the interval $180 \leq n \leq 200$ where $n$ is the iteration number

time series $x_{11}^{(n)}$, $x_{12}^{(n)}$, $x_{21}^{(n)}$, $x_{22}^{(n)}$ and the appropriate eigenvalues in the interval $0 \leq n \leq 50$, but the processes calm down at a higher $n$. Particularly, Fig. 6C shows that iterated matrices become scalar matrices. Fig. 6D shows that eigenvalues $\lambda_0^{(n)}$ oscillate in the interval between 0 and 1 which is a necessary (but not a sufficient) condition of convergence of the product in (53). It is of interest to note that variable $\mu_0^{(n)}$ tends to zero thus ensuring the boundedness of $\{x_{kl}^{(n)}\}_{n=0}^{+\infty}$; $k, l = 1, 2$. A different value of parameter $a$ ($a = 3.6$) yields a violent divergence of iterative processes (Fig. 7).

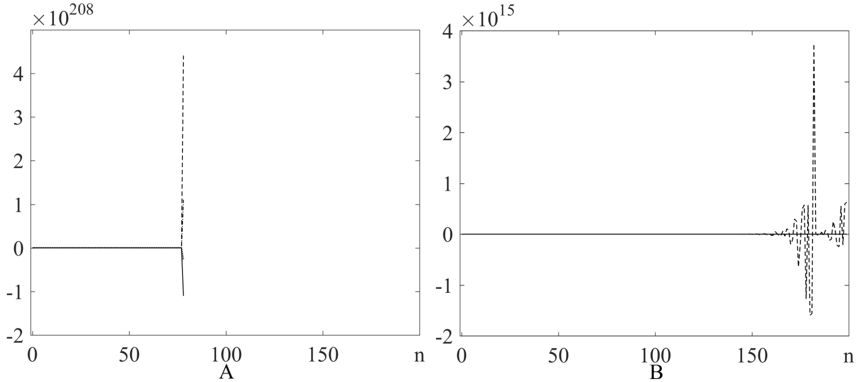


**Figure 7.** The evolution of the logistic map of matrices from $\mathbf{X}^{(0)} = \begin{bmatrix} 1.3 & 4 \\ -0.25 & -0.7 \end{bmatrix}$ at $a = 3.6$ (A showing the evolution of $x_{11}^{(n)}$, $x_{12}^{(n)}$, $x_{21}^{(n)}$, $x_{22}^{(n)}$; B showing the evolution of the eigenvalue (——) and the parameter $\mu_0^{(n)}$ (----) defined by Eq. (53))

*The sensitivity to the initial conditions at $a = 3.7$.* It is well-known that a scalar logistic map evolves to chaos after a cascade of period doubling bifurcations.

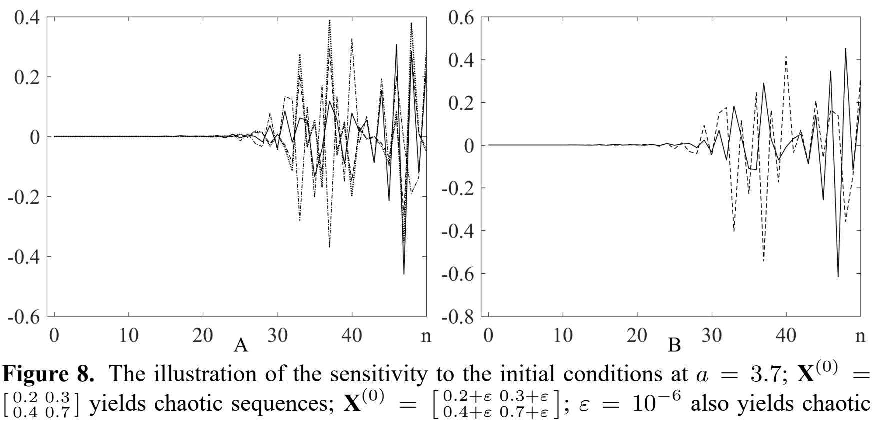


**Figure 8.** The illustration of the sensitivity to the initial conditions at $a = 3.7$; $\mathbf{X}^{(0)} = \begin{bmatrix} 0.2 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$ yields chaotic sequences; $\mathbf{X}^{(0)} = \begin{bmatrix} 0.2+\varepsilon & 0.3+\varepsilon \\ 0.4+\varepsilon & 0.7+\varepsilon \end{bmatrix}$; $\varepsilon = 10^{-6}$ also yields chaotic sequences; A shows differences between the appropriate elements of matrices; B shows differences between the appropriate eigenvalues

At $a = 3.7$ the dynamics of a scalar logistic map is already chaotic. The sensitivity to the initial conditions is one of the characteristic features of the deterministic chaos (Strogatz, 2014). We will illustrate this feature by using the logistic map of matrices.

The matrix of initial conditions $\mathbf{X}^{(0)} = \left[\begin{smallmatrix} 0.2 & 0.3 \\ 0.4 & 0.7 \end{smallmatrix}\right]$ and its eigenvalues yield chaotic sequences at $a = 3.7$. We construct a perturbed matrix of initial conditions $\mathbf{X}^{(0)} = \left[\begin{smallmatrix} 0.2+\varepsilon & 0.3+\varepsilon \\ 0.4+\varepsilon & 0.7+\varepsilon \end{smallmatrix}\right]$; $\varepsilon = 10^{-6}$ and follow the iterative processes. Differences between the values of the iterated elements and the iterated eigenvalues of these matrices are shown in Fig. 8A and Fig. 8B.

### 3.3. Self-induced resonance on the iterative map of matrices

The effect of self-induced resonance in generalized iterative maps of matrices can be defined and described by using formal algebraic techniques (Navickas et al., 2011). The effect of self-induced resonance can be observed in an iterative map of square matrices of order 2 if and only if the matrix of the initial conditions is a nilpotent matrix and the Lyapunov exponent of the corresponding scalar iterative map is greater than zero. Computational experiments with the logistic map and the circle map are used to illustrate the effect of self-induced resonance occurring in iterative maps of matrices (Navickas et al., 2012).

**Definition 11.** Self-induced resonance occurs in the iterative map of matrices if

$$\lim_{n \to +\infty} \prod_{k=0}^{n} \left| f'\left(\lambda_0^{(k)}\right) \right| = +\infty, \tag{59}$$

and the eigenvalue of the iterative nilpotent matrix remains bounded $\left|\lambda_0^{(n)}\right| \leq M < +\infty$; $n = 0, 1, 2, \ldots$.

**Remark.** Definition 11 implies that the effect of self-resonance in the iterative map of matrices cannot be observed if the matrix of the initial conditions is an idempotent matrix.

**Conclusion 9.** Self-induced resonance occurs in an iterative map of matrices if the matrix of the initial conditions is a nilpotent matrix and the Lyapunov exponent of the corresponding scalar iterative map is greater than zero.

The Lyapunov exponent of a scalar map of matrices reads (Hilborn, 2000):

$$\tilde{\lambda} = \frac{1}{n} \sum_{j=0}^{n-1} \ln \left| f'\left(\lambda_0^{(j)}\right) \right|; \tag{60}$$

where $\tilde{\lambda}$ is a numerical estimate of the Lyapunov exponent. Lyapunov exponents are calculated for a sequence of iterative values $\lambda_0^{(j)}$ when all the transient processes have ceased down.

*Computational results*. It has been shown in the previous section that the effect of self-induced resonance occurs in the iterative map of matrices if the matrix of the initial conditions is a nilpotent matrix and the Lyapunov exponent of the corresponding scalar iterative map is greater than zero. We will perform computational experiments with the logistic map and the circle map in order to illustrate these theoretical results.

Fig. 9 illustrates the evolution of the circle map of matrices when the matrix of the initial conditions is an idempotent matrix. It is clear that the effect of resonance cannot be observed in such a system; both eigenvalues of the iterative matrix are locked in the 3:7 mode. The effect of self-resonance cannot be observed in the evolution of the logistic map of matrices when the matrix of the initial conditions is an idempotent matrix even though the dynamics of the corresponding scalar map is chaotic (Fig. 10; the Lyapunov exponent of the scalar iterative map is equal to $0.4312$).

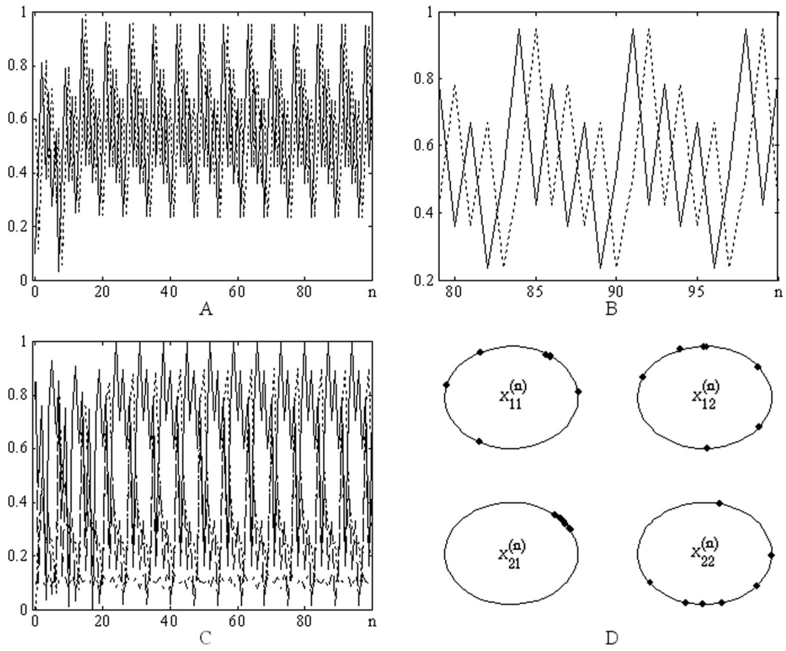


**Figure 9.** The circle map of matrices does not exhibit the effect of resonance when the matrix of the initial conditions is an idempotent matrix ($\alpha = 2, \beta = 8$, $\lambda_1^{(0)} = 0.1$, $\lambda_2^{(0)} = 0.6$); parameters $K = 0.96$ and $\Omega = 0.428$ result in 3:7 synchronization. A shows the evolution of eigenvalues in the interval $0 \leq n \leq 100$ ($n$ is the iteration number); B is the zoomed image of A in the interval $80 \leq n \leq 100$. C shows the evolution of $x_{11}^{(n)}$, $x_{12}^{(n)}$, $x_{21}^{(n)}$, $x_{22}^{(n)}$ whereas D shows the phases of each element of the iterative matrix after transient processes have ceased down.

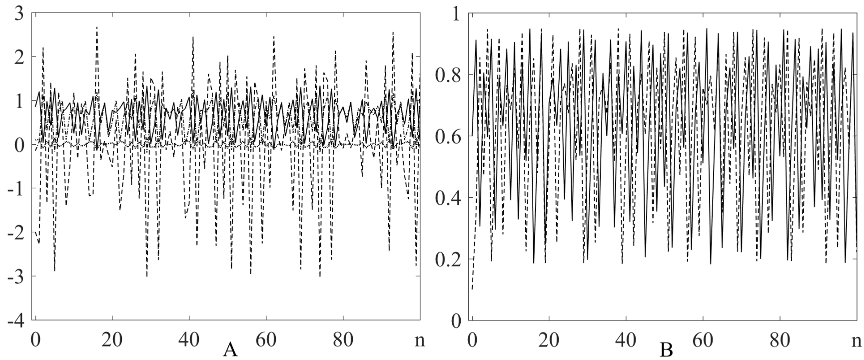


**Figure 10.** The logistic map of matrices does not exhibit the effect of resonance when the matrix of the initial conditions is an idempotent matrix ($\alpha = 2, \beta = 8$, $\lambda_1^{(0)} = 0.1$, $\lambda_2^{(0)} = 0.6$); parameter $a = 3.8$. A shows the evolution of $x_{11}^{(n)}$, $x_{12}^{(n)}$, $x_{21}^{(n)}$, $x_{22}^{(n)}$; B shows the evolution of eigenvalues in the interval $0 \leq n \leq 100$ ($n$ is the iteration number)

Fig. 11 shows the evolution of the logistic map of matrices when the matrix of the initial conditions is a nilpotent matrix. The initial tendency to resonate can be observed until the transient processes have ceased down. The Lyapunov exponent of the iterative map

$$\lambda_0^{(n+1)} = a\lambda_0^{(n)} \left(1 - \lambda_0^{(n)}\right); \tag{61}$$

is equal to $-0.8723 < 0$ (at $a = 3.5$). Thus, in the long run, the logistic map of matrices quiets down.

Fig. 12 illustrates the effect of self-induced resonance in the logistic map of matrices. Now, the Lyapuvov exponent of Eq. (61) is equal to $0.4312 > 0$ (at $a = 3.8$), and the system experiences a violent divergence (computations are

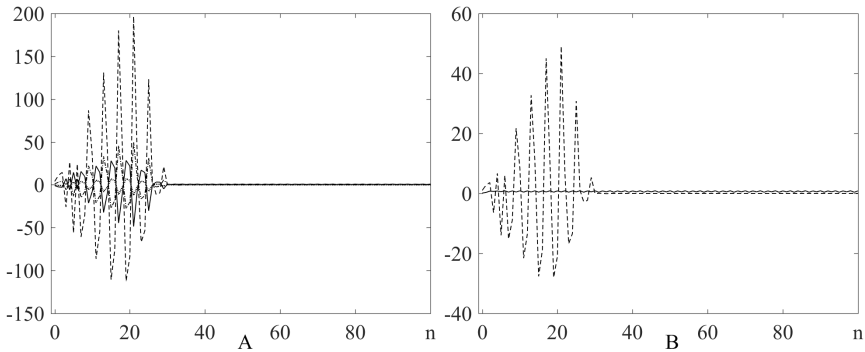


**Figure 11.** The logistic map of matrices shows an initial tendency to resonate but quiets down when the transient processes cease down. The matrix of the initial conditions is a nilpotent matrix ($\alpha = 2, \beta = 8$, $\lambda_0^{(0)} = 0.1$); $a = 3.5$. A shows the evolution of $x_{11}^{(n)}$, $x_{12}^{(n)}$, $x_{21}^{(n)}$, $x_{22}^{(n)}$; B shows the evolution of the eigenvalue (——) and parameter $\mu_0^{(n)}$ (----)

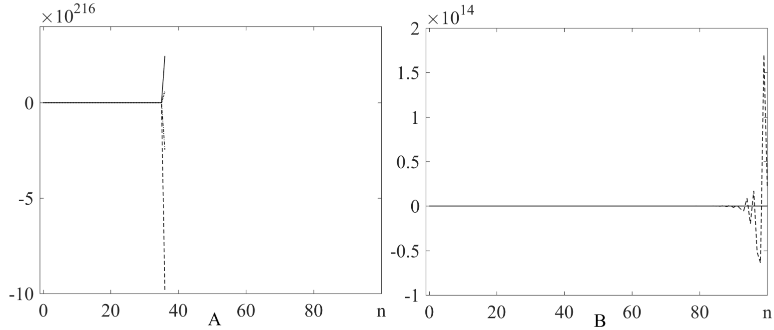


**Figure 12.** The logistic map of matrices exhibits the effect of self-induced resonance. The matrix of the initial conditions is a nilpotent matrix ($\alpha = 2, \beta = 8$, $\lambda_0^{(0)} = 0.1$); $a = 3.8$. A shows the evolution of $x_{11}^{(n)}$, $x_{12}^{(n)}$, $x_{21}^{(n)}$, $x_{22}^{(n)}$; B shows the evolution of the eigenvalue (——) and parameter $\mu_0^{(n)}$ (----)

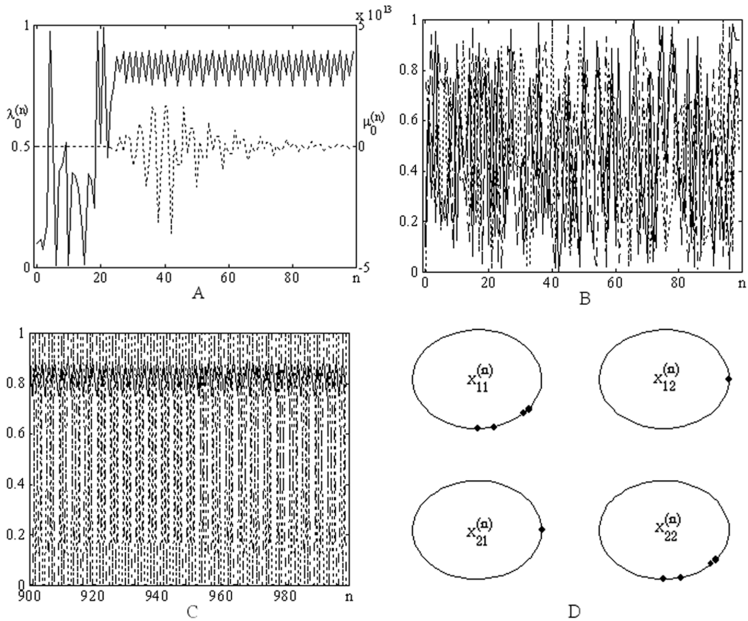


**Figure 13.** The circle map of matrices shows an initial tendency to resonate but quiets down when the transient processes have ceased down. The matrix of the initial conditions is a nilpotent matrix ($\alpha = 2, \beta = 8$, $\lambda_0^{(0)} = 0.1$); $K = 4.4$ and $\Omega = 0.428$. A shows the evolution of the eigenvalue (——) and parameter $\mu_0^{(n)}$ (----). B shows the evolution of $x_{11}^{(n)}$, $x_{12}^{(n)}$, $x_{21}^{(n)}$, $x_{22}^{(n)}$. C and D illustrate the evolution of the system after the transient processes have ceased down (elements of the iterative matrix are shown in C and the phases of elements of the iterative matrix are shown in D)

terminated due to the numerical overflow). Analogously, the quieting of the circle map of matrices is illustrated in Fig. 13; the Lyapunov exponent of the iterative map

$$\lambda_0^{(n+1)} = \lambda_0^{(n)} + \Omega - \frac{K}{2\pi} \sin\left(2\pi\lambda_0^{(n)}\right) \tag{62}$$

is equal to $-0.0546 < 0$ (at $K = 4.4$ and $\Omega = 0.428$). On the other hand, the effect of self-induced resonance is observed in the circle map of matrices at $K = 4.6$ and $\Omega = 0.428$ (Fig. 14); the Lyapunov exponent of the scalar iterative map Eq. (62) is equal to $0.4208 > 0$ in this case.

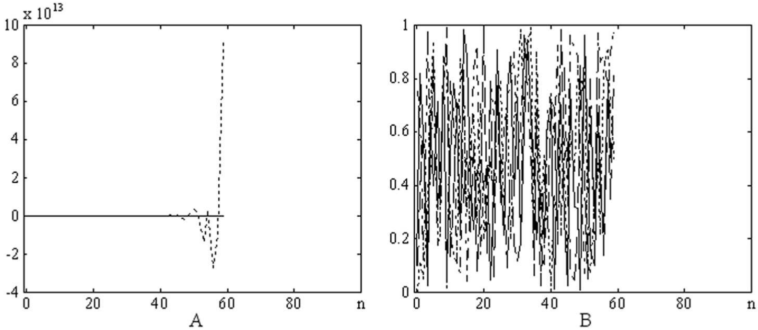


**Figure 14.** The circle map of matrices exhibits the effect of self-induced resonance. The matrix of the initial conditions is a nilpotent matrix ($\alpha = 2, \beta = 8, \lambda_0^{(0)} = 0.1$); $K = 4.6$ and $\Omega = 0.428$. A shows the evolution of the eigenvalue (——) and parameter $\mu_0^{(n)}$ (- - - -); B shows the evolution of $x_{11}^{(n)}$, $x_{12}^{(n)}$, $x_{21}^{(n)}$, $x_{22}^{(n)}$

### 3.4. Concluding Remarks

The iterative maps of the matrices of order 2 are described in this section. The main dynamical features of iterative maps of matrices are discussed and illustrated with numerical examples when the function of the iterative map is analytical. The effect of self-induced resonance in generalized iterative maps of matrices is described in this section. Necessary and sufficient conditions for the existence of such resonance are derived and illustrated with computational experiments. So far, we have investigated the maps of the square matrices of order 2 only; yet it must be noted that order 2 matrices already allow interesting generalizations. Iterative maps of a higher order and concrete applications where the effect of self-induced resonance can be exploited, for example, as a factor ensuring the security of the encoding scheme, are definite objects of interest for future research.

## 4. ANALYSIS OF ELECTROCARDIOGRAPHIC PARAMETERS

Human body is a complex system. Thus there is no use in the analysis of separate body system components as the complex system consists of many components which interact with each other. Interdependencies between systems should be analyzed instead.

The methodology to investigate the relations between two human body systems is analyzed in this study. This methodology is based on matrix special structural decomposition. The importance of such an investigation and the classification of the second order matrices was introduced in Section 2.4. Section 3 emphasized that the type of matrix of the initial conditions in a chaotic iterative map may result in explicitly different processes (Navickas et al., 2011; Navickas et al., 2012).

Human body is a great example of nonlinear dynamical systems. Therefore, the analysis of interrelations in human body is related to the analysis of the matrix structure. The structure of a square matrix may be defined by using discriminants, matrix trace, eigenvalues, etc. (Sec. 2.3.3). Thus one of the tasks of this study is to introduce several matrix structure-related characteristics and demonstrate the results on real medical data.

### 4.1. Analysis of relation dynamics between two synchronized signals

Most methods used to evaluate the relation between two signals require extensive data sets and the momentum information is lost. The proposed methodology requires only three data points (past, present and future) to obtain a coefficient related to the matrix structure at moment $n$ and may be used as a real-time signal analyzer.

*Electrocardiographic data.* The lead where the changes are most obvious should be investigated for people with cardiac pathology, e.g. acute myocardial infarction (Jeon et al., 2014). The choice of ECG parameters is based on the phenomenological model where human is a complex system (Fig. 1). ECG parameters used in this research are: RR (ms), DJT (ms), DQRS (ms), AR ($\mu$V), AT ($\mu$V).

*Sequence of matrices.* A matrix of order 2 was constructed in a particular way. The special matrix form emphasizes the fact that human body is more sensitive to the change of affect than to the absolute value (e.g. temperature, light intensity change). Therefore, the matrix was constructed by using first order Lagrange differences. Matrix elements are constructed by using two synchronized data sets. In order to construct $n$th matrix, three data points $n-1$, $n$ and $n+1$ are required.

Let two data sequences be given $Y = (y_0, y_1, y_2, \ldots)$ and $Z = (z_0, z_1, z_2, \ldots)$. Then the sequence of matrices $\mathbf{X}_n$ is constructed:

$$\mathbf{X}_n := \mathbf{X}_n(Y, Z) = \begin{bmatrix} y_n & y_{n-1} - z_{n-1} \\ y_{n+1} - z_{n+1} & z_n \end{bmatrix}; \tag{63}$$

where $n = 1, 2, \ldots$. Data sequences $Y$ and $Z$ represent different ECG parameters that may differ not only in scale but also in measure units. Therefore, the data is normalized to interval $[0,\ 1]$ (Etzkorn, 2012):

$$\hat{y}_n = \frac{y_n - y_{\min}}{y_{\max} - y_{\min}}; \tag{64}$$

where $n$ – cardiocycle number; $y_n$ – real parameter value in cardiocycle $n$; $y_{\min}$ and $y_{\max}$ – physiological limits of the parameter.

*Three component matrix decomposition for ECG analysis*. The importance of the matrix structure is discussed in Sections 2 and 3. Thus each matrix $\mathbf{X}_n$ from (63) can be decomposed by using Eq. (28). If slightly different parametrization is chosen $\alpha = \frac{1}{2}, \beta = 2\beta_0$, then (28) takes the form:

$$\mathbf{X} := \frac{x_{11} + x_{22}}{2} \cdot \mathbf{I} + \frac{\sqrt{\operatorname{dsk}\mathbf{X}}}{2} \begin{bmatrix} 0 & \beta_0 \\ \frac{1}{\beta_0} & 0 \end{bmatrix} + \frac{x_{11} - x_{22}}{2} \begin{bmatrix} 1 & \beta_0 \\ -\frac{1}{\beta_0} & -1 \end{bmatrix}; \tag{65}$$

where $\beta_0 = \frac{-(x_{11}-x_{22})+\sqrt{\operatorname{dsk}\mathbf{X}}}{2x_{21}} = \frac{2x_{12}}{\sqrt{\operatorname{dsk}\mathbf{X}}+(x_{11}-x_{22})}$, ($\sqrt{\operatorname{dsk}\mathbf{X}}$ has two values). Let us note that the following relations hold true: $\lambda_k = \frac{1}{2}\left(\operatorname{Tr}\mathbf{X} + \sqrt{\operatorname{dsk}\mathbf{X}}\right)$ $(k = 1, 2)$, $\operatorname{Tr}\mathbf{X} = x_{11} + x_{22} = \lambda_1 + \lambda_2$, $\frac{\sqrt{\operatorname{dsk}\mathbf{X}}}{2} = \frac{\lambda_1 - \lambda_2}{2}$.

Therefore, values $\operatorname{dsk}\mathbf{X}$, $\lambda_1$, $\lambda_2$,$\beta_0$,$\operatorname{Tr}\mathbf{X}$ contain information about the structure of matrix $\mathbf{X}$ and require further investigation.

## 4.2. Greco-Roman wrestlers' data analysis

Every sportsman has to go through a medical examination regularly. Medical examination includes not only blood or urine tests but also a variety of physical tests where the physical capabilities are assessed. In this study, we used the data obtained from one of such tests – the Ruffier test. Changes in the function of body systems were investigated in this section (Smidtaite et al., 2009; Smidtaite et al., 2010; Šmidtaitė et al., 2009).

Data of fourteen Lithuanian sportsmen was analysed. The sportsmen's age varied from 22 to 26 years, height 180 cm ($\pm$30cm), weight 90 kg ($\pm$10kg). Each sportsman performed the Ruffier physical test which consists of 30 squats per 45 seconds. The ECG was recorded during the three test stages: before the physical load (1 min), during the Ruffier test and finally, after 2 min of recovery (divided into the beginning (1 min) and the end (1 min) of recovery) (Fig. 15).

Multivariate signal analysis offers many methods to evaluate the relation between the two signals. One of the most classical and oldest methods is the correlation function. *Spearman's correlation coefficient* $r_S$ was chosen because no certain data distribution is required. The main problem is that this method requires

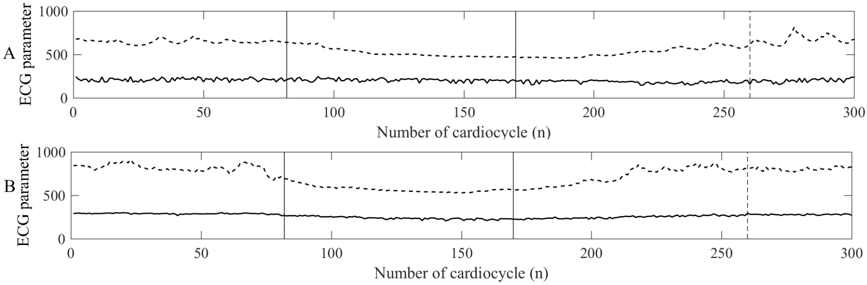


**Figure 15.** Duration of RR (solid line) and DJT (dashed line) in each cardiocycle is shown for less than 5-year experienced wrestler in part A. Part B shows the data for a more than 5-year experienced sportsman

a rather big amount of data points, therefore, the result is very abstract. Spearman's correlation coefficient $r_S$ is calculated in a sliding window of 20 points. Figure 16 shows the sequence of Spearman's correlation coefficients for the examined sportsmen. Spearman's correlation coefficients vary during all the test stages, and no obvious differences either between two sportsmen or among different test stages can be seen.

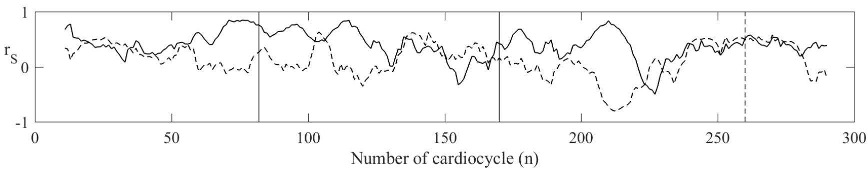


**Figure 16.** Spearman's correlation coefficients for RR and DJT parameters. The solid line denotes the correlation coefficients for a more than 5-year experienced wrestler, the dashed line denotes the correlation coefficients for a less than 5-year experienced wrestler

*Granger causality* for RR and DJT parameters was investigated in this study[1]. The Granger causality test was performed for a sliding window of 20 observations in order to investigate the dynamics of processes. Although the Granger causality test requires a larger sample size but in practice, especially in econometric calculations, the nature of the data compels the researcher to work with a smaller sample size. For each sliding window, the value of $F$-statistic is evaluated as well as the critical value $c_F$ from the $F$-distribution ($\alpha = 0.05$). If $F > c_F$ then the hypothesis (that there is Granger cause between the two observed processes) is not rejected.

The Granger causality test results for two wrestlers with different training experience are shown in Fig. 17. The ordinate represents the difference $F - c_F$

[1]Function granger_cause.m was used in this study. The code was developed by Dr. Chandler Lutz (University of California Riverside, USA, 2009). Website https://se.mathworks.com/matlabcentral/fileexchange/25467-granger-causality-test

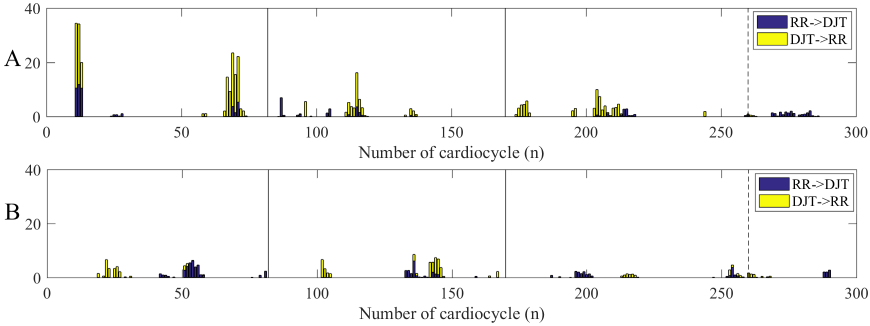


**Figure 17.** The ordinate represents $F - c_F$ (if $F - c_F > 0$ the Granger causality exists). Part A shows a more than 5-year experienced wrestler's data; Part B shows a less than 5-year experienced wrestler's data

(when $F - c_F > 0$ the Granger causality exists). Figure 17A shows Granger causality for a wrestler with more than 5-year training experience (data stationarity: 95.9% (RR) and 100% (DJT)). Granger causality (RR→DJT or DJT→RR) was observed in 30% of the data. Figure 17B shows Granger causality test results for a less experienced sportsman (data stationarity: 90.9% (RR) and 100% (DJT)). The test showed that 29.3% of RR and DJT data for a less experienced wrestler has Granger causality. Stationarity of processes were tested by using augmented Dickey–Fuller test (used a MATLAB function *adftest*). The Granger's causality test results for both sportsmen (more than 5-year and less than 5-year experience) seem to be rather random, without showing any tendencies at any different stages of the test.

*Structural matrix analysis based methodology* observes the characteristics related to the matrix structure. One of such characteristics is discriminant dsk $\mathbf{X}$. The sequence of discriminants dsk $\mathbf{X}_n$ for parameters RR and DJT for the two wrestlers with different training experience is shown in Fig. 18. Body systems remain stable during the rest stage (minor fluctuations). The relation between the systems decreases (the discriminant increases) when the physical load occurs. The relation parameter is the reciprocal of the discriminant value. Discriminant values decrease during the recovery stage. The most significant difference between the discriminants for the two wrestlers occur when a physical load is applied.

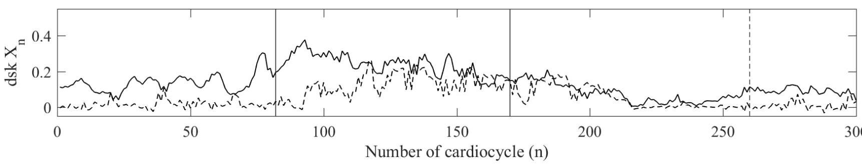


**Figure 18.** The sequence of discriminants for parameters RR and DJT: the solid line denotes a more experienced wrestler's data whereas the dashed line denotes a less experienced wretsler's data

It can be noted that a more sudden increase in discriminant value during the physical load stage occurs for a more experienced sportsman rather than for a less experienced wrestlers (Fig. 18). The discriminants and bounded lines are shown in Fig. 19.

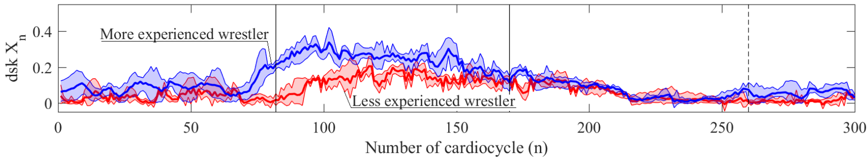


**Figure 19.** Sequence of discriminants for RR and DJT parameters for all the 14 sportsmen

Phase portraits are an invaluable tool in studying dynamical systems. The phase portrait is a highly convenient geometric representation of the trajectories of a dynamical system. Phase portrait shows homogeneity of repolarization processes of myocardium when the heart works intensively during the physical load (Alabdulgade et al., 2015; Venskaitytė et al., 2010). The results showed that a different test stage forms a different attractor. When body systems go to a different stage, bifurcations occur, and a new stable state is formed (Fig. 20).

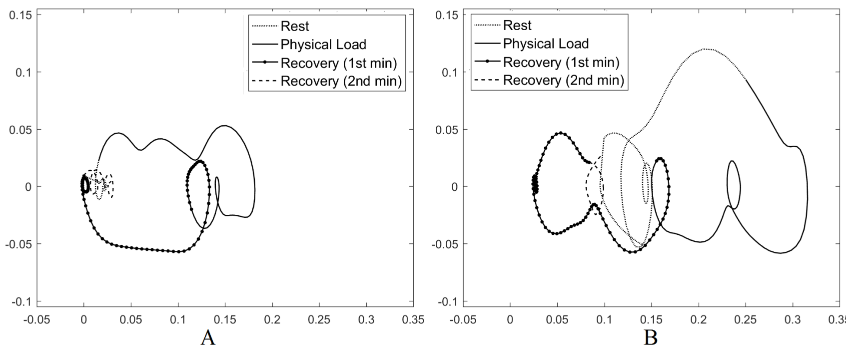


**Figure 20.** The phase portrait of the discriminant for RR and DJT is shown. The abscissa represents dsk $\mathbf{X}_n$ and the ordinate represents the derivative of discriminant. A shows the discriminant's phase portrait for a less experienced wrestler and B shows a more experienced sportsman's data

Different phase portraits were drawn due to the changes in the physical load and different sportsmen's training experience (Fig. 20). The discriminant for RR and DJT parameters is more expressed for the sportsmen with more than 5 years of experience. It is related to the increased activity of the sympathetic nervous system that was caused by the increased metabolic activity. A more experienced wrestler's body regulatory and supplying systems are better adapted to the physical load; therefore, regulatory processes and the following perturbations occur faster than in the less experienced sportsmen's body. The recovery stage attractor contains lower discriminant values than the beginning (Rest) stage attractor for the more experienced sportsmen. It means that the body recovers completely after the physical load and that no residual fatigue is detected. The attractors in the recovery and rest stages do not coincide for a less experienced wrestler. Body systems are not able to return (in the two minutes after the load is over) to the same state as it was before the load because the residual fatigue occurs.

The dynamics of eigenvalues are different from that of discriminants. It should be noted that eigenvalues classify wrestlers in two separate groups and that no common area is seen (see Fig 21). Traces of matrices enables us to divide the test into separate stages: trace decreases for both wrestler groups in the load stage, the trace begins to increase after the physical load is over and in the 2nd minute of the recovery converge to the initial trace values (that were during the rest stage) (see Fig. 22). Traces form the two separate lines for two wrestler groups (although there are some common areas).

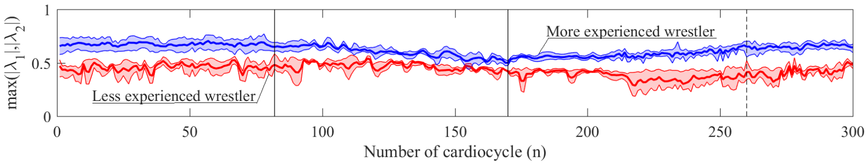


**Figure 21.** Sequences of matrix eigenvalues for RR and DJT parameters for all the 14 sportsmen

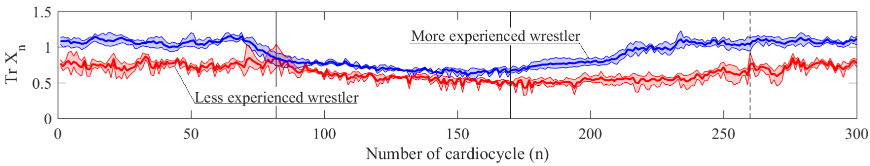


**Figure 22.** Sequences of the matrix trace for RR and DJT parameters for all the 14 sportsmen

It should be noted that the Ruffier test is low activity if compared to tests where the physical load is applied increasingly till the signs of fatigue occur. Although the physical load is small in the Ruffier test but the structural matrix based methodology was able to identify the sportsmen with better hemodynamics in the heart muscles, which is related to better training abilities.

### 4.3. Coronary angioplasty data

Urgent treatment is necessary when sections of the heart (coronary) arteries are narrowed. It is natural that the effect more or less is felt in the whole body systems. During such a procedure, it is important to observe how one system responds to the functional changes in another system. RR and DQRS parameters were analyzed in this study in order to observe the relations between the heart and human body regulation. Figure 23A shows raw data of ECG parameters for a patient during the coronary angioplasty procedure when a catheter was inserted into a blood vessel in the groin. Electrocardiographic data observation consists of three stages: before, during and after the procedure.

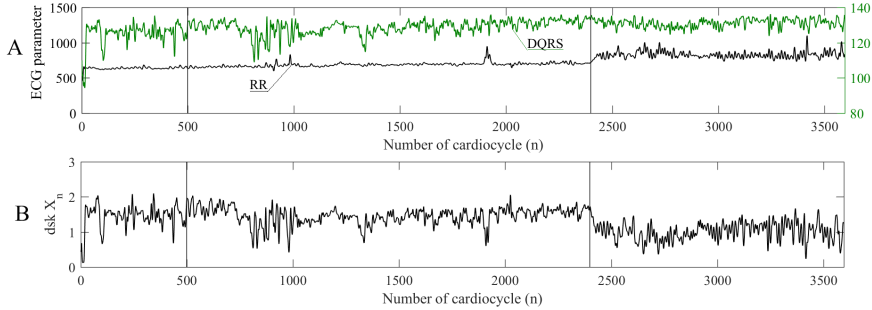


**Figure 23.** Raw data is shown in part A. The observation consists of three stages: before, during and after the coronary angioplasty. Part B shows the discriminants for RR and DQRS

Human body encounters stress when the blood circulation is impaired. The balance in cooperation of human body systems is disrupted (one system dominates against another, etc.). When the blood flow is restored (the last stage of the procedure), the balance of human body systems recovers (the values of discriminants decrease) (see Fig. 23B).

Most significant changes in the mean and slope of discriminant values detected by MATLAB function *findchangepts* are shown (see Fig. 24). First significant change is observed slightly later (about 730th cardiocycle) than the invasive procedure started (about 500th cardiocycle)) which is understandable. The end of the procedure is determined exactly by MATLAB tools applied to the sequence of discriminants. Spearman's correlation coefficient was calculated for each twenty sample sliding window. The sequence of Spearman's correlation coefficients are shown in Fig. 25. Correlation coefficients fluctuate during the whole period from $-1$ to 1 and no obvious changes among stages can be detected.

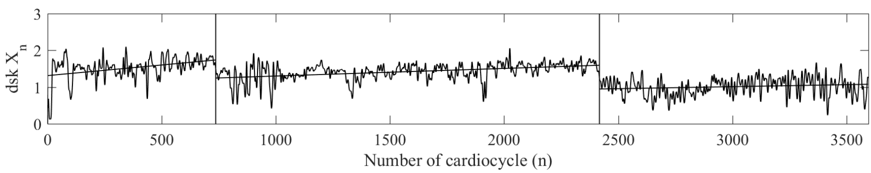


**Figure 24.** Discriminant changes found with MATLAB function *findchangepts*

Invasive procedures are always difficult to predict. The time of coronary revascularization, reaction to the procedure and complications (e.g. reperfusion arrhythmia) for each patient are different and unique, therefore, they require further analysis.

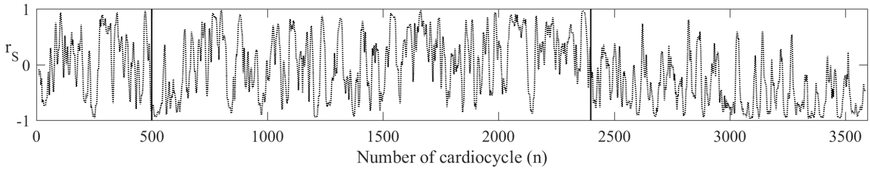


**Figure 25.** Spearman's correlation coefficient between parameters RR and DQRS with a sliding window of 20 samples

### 4.4. Heart rate variability and Earth local magnetic field relation analysis

Earth's magnetic field, also known as geomagnetic field, extends from Earth's interior out into space, where it meets the solar wind. The magnitude at the Earth's surface ranges from 25 to 65 microteslas. Thus it is not surprising that a question "how do fluctuations of the Earth's magnetic field affect a human being?" arise (Alabdulgade et al., 2015).

Heart-rate variability data from 17 female volunteers was collected during a long-term project from April 1, 2012, to August 31, 2012. All the participants were employees of the Prince Sultan Cardiac Center in Hofuf Saudi Arabia (7 nursing staff, 6 housekeeping and 4 from the research department). The average age was 32±8 years, ranging from 24 to 49 years. Two participants experienced skin irritation from ECG electrodes. Therefore, they dropped out of the study. The participants signed the informed consent form before taking part in the study and were free to withdraw from the experiment at any time.

All the participants underwent weekly 24–72 hour ambulatory HRV recordings with Firstbeat Bodyguard HRV recorders; the Bodyguard HRV recorder calculates the RR interval from the ECG measured at 1000 samples per second. Participant recordings were generally 72 hours in length and scheduled once a week over a 5 month period between April and the end of August 2012. The mean RR was calculated for every hour in the recording and the time was synchronized with the local hourly mean magnetic field (B) measurements (Hofuf, Saudi Arabia).

Geomagnetic activity affects people in different ways. For some people it may have no effect and for some the magnetic field may even cause serious health disorders. The coherence between the RR intervals (ms) and the local magnetic field (picoteslas, pT) was analysed in this study. The relations were observed with the help of the discriminant value and, later, the sensitivity coefficient $S$ was introduced. Sensitivity coefficient $S$ evaluates the relation between the local magnetic field $\tilde{\mathrm{B}}$ and discriminants $\operatorname{dsk}\mathbf{X}_n(\tilde{\mathrm{B}}, \mathrm{RR})$:

$$\tilde{\mathrm{B}} = S \cdot \operatorname{dsk}\mathbf{X}_n(\tilde{\mathrm{B}}, \mathrm{RR}) + b. \tag{66}$$

Healthy human body is able to adapt to the changing magnetic field. One of such cases is shown in Fig. 26. Figure 26C shows how the relations between the Earth's local magnetic field and heart rate variability change when the magnetic field changes. Sensitivity coefficient $S$ is positive when a human being does not have serious health issues (see Fig. 27).

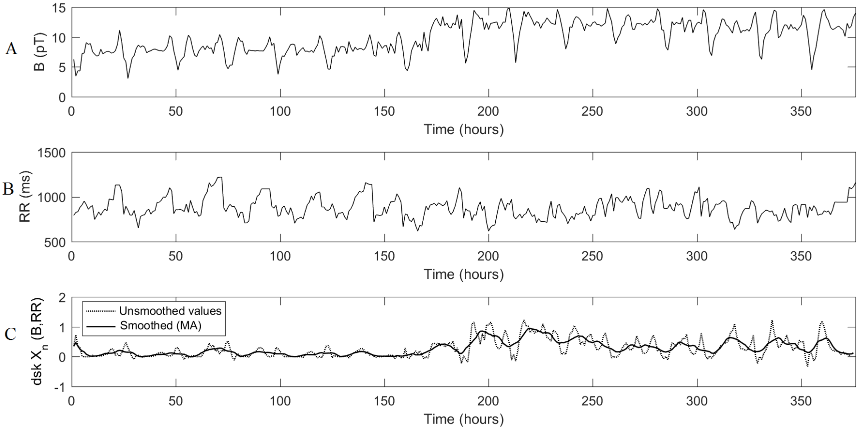


**Figure 26.** A shows the local Earth's magnetic field (pT); B shows RR intervals (ms). C illustrates discriminants between the magnetic field and RR intervals

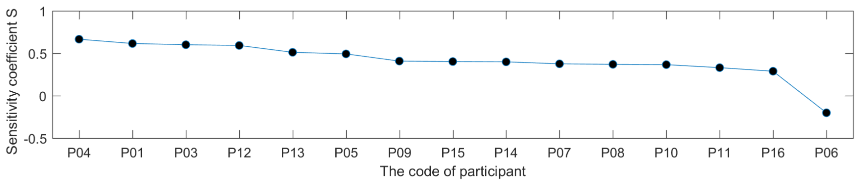


**Figure 27.** Heart Rhythm sensitivity to Earth local magnetic field fluctuations in the descending order for all the investigated persons

The ability to adapt to magnetic field fluctuations is not common for people who have health issues. The ability to adapt to magnetic field fluctuations varies from person to person, and it is normal because it is almost impossible to find a completely healthy person. But sometimes the disability of an organism to adapt to the changing environment is critical and may increase the already existing or cause new health disorders. The described situation is shown in Figure 27 (participant P06). According to case record, participant P06 has serious cardiac problems which was also revealed by sensitivity coefficient $S$.

Adaptivity to the changing environment, including magnetic field fluctuations, is an important health indicator, and when this organism ability decreases, it may cause serious health issues and outcome. When major magnetic fluctuations occur, every organism encounters stress which may provoke elevation of blood pressure or even myocardial infarction (Alabdulgade et al., 2015).

### 4.5. Conclusions

Complex system analysis emphasizes the great importance of interrelations between system parts. Statistical methods require large amounts of data and are not convenient when momentum information is needed. Therefore, methodology based on structural matrix analysis reveals new possibilities. In this chapter, characteristics related to the matrix structure are analyzed: discriminants, eigenvalues, trace. The observation of the change in dynamics of these characteristics during the physical load test helped to compare the physical fitness of different sportsmen groups. Dynamics of discriminants during the different stages of the angioplasty procedure revealed different levels of cooperation of human body systems in a stress and health disorder situation. Discriminants allowed to identify the beginning of the improved blood flow after the procedure. Finally, structural matrix analysis based methodology was introduced in other than ECG signal analysis. Observation of discriminants and sensitivity coefficients enabled to evaluate the human sensitivity to the local geomagnetic field and identify participants with serious health disorders.

## CONCLUSIONS

1. Definitions of simple and compound idempotents and nilpotents were introduced. The set of idempotents and nilpotents and their parametric expressions were defined. A scheme was developed how to generate the system of idempotents and nilpotents.
2. *Special* three component decomposition of second order matrices was introduced. The formulas to calculate coefficients for decomposition were constructed. Special matrix decomposition enabled to simplify formula for matrix $n$ power $\mathbf{X}^n$. The necessary and sufficient conditions were derived for two matrices to commute.
3. *Modified* class of iterative maps of matrices $\mathbf{X}^{(n+1)} = f(\mathbf{X}^{(n)})$ where $\mathbf{X}^{(n)} \in \mathbb{R}_{2\times 2}$ and $f$ is an analytical function. The necessary and sufficient conditions were derived for the divergence of such iterative maps. The modified class of iterative maps exhibits effects that are typical neither to extensions of scalar iterative maps nor to coupled map lattices.
4. *Novel* estimates based on structural matrix analysis were introduced for evaluation of two signal coherence. Proposed characteristics enabled to draw tendencies in coherence dynamics for two groups of sportsmen with different training experience. The observation of interrelations of RR and DQRS parameters helped to identify the beginning of improvements in blood circu-

lation during coronary angioplasty procedure. The investigation of discriminants and sensitivity coefficients enabled to identify health disorders.

5. Coherence investigation tool based on structural matrix analysis was integrated in the automatic ECG analysis system "Kaunas-Load".

## PRACTICAL IMPLICATIONS

1. 2008–2012 m. two EUREKA projects were performed: "A guardian angel for the extended home environment" ITEA 2 GUARANTEE 08018 and "Assessment system of distributed intelligence functional status for the elderly and disabled" EUREKA E!4452 EDFAS.
2. Proposed ECG analysis methodology was used in project "Methods and systems for predicting of acute hypotensive episodes" (Nr. VP1-3.1-ŠMM-10-V-02-003) (2013-2015).
3. 2014–2015 m. project was performed "Research on the relations between Earth's magnetic field, human's and animal's cardiovascular systems (GEOMAG)".
4. 2014 m. applied for HORIZON 2020 – "Local geomagnetic field fluctuations impact on human and animal health and their cardiovascular system functional state" but the application was rejected.
5. Currently (beginning in 2014) international project "Global Coherence Initiative" (leader country USA, California, HeartMath Institute). Participants from Lithuania: Lithuanian University of Health Sciences and Kaunas University of Technology.
6. The *Original* methodology developed during project "Methods and systems for predicting of acute hypotensive episodes" (2013–2015) for ECG analysis was used in three patents.

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## ANTROS EILĖS MATRICŲ STRUKTŪRINIAI DĖSTINIAI NETIESINIŲ SISTEMŲ TYRIMUOSE

### Temos aktualumas

Nors netiesinių dinaminių sistemų tyrimų užuomazgos, siejamos su J. H. Poincare darbais, siekia XIX a., tačiau tokios sistemos vis dar yra svarbus ir didelio dėmesio sulaukiantis tyrimų objektas. Netiesinės dinaminės sistemos sutinkamos daugelyje mokslo sričių: matematikoje, fizikoje, biologijoje, ekonomikoje ir netgi psichologijoje. Dauguma mus supančių realių sistemų yra sudėtingos (kompleksinės), o jas apibūdinantys signalai – daugiakanaliai. Tokių signalų pavyzdžiai yra ne tik biomedicininiai, bet ir seismologiniai, lazerių ir kiti technikos ar gamtoje generuojami signalai. Nagrinėjant kompleksines sistemas, pavienių signalų tyrimas nėra pakankamas visos sistemos dinamikai aprašyti, tuomet reikia vertinti signalų vidinius bei tarpusavio sąryšius.

Dviejų signalų tarpusavio ryšiui vertinti dažniausiai taikomi standartiniai statistiniai metodai, tokie kaip koreliacija, kroskoreliacija, Grangerio priežastingumo testas, abipusė informacija ir kt. Statistinių dydžių skaičiavimams reikia sąlyginai daug duomenų, be to, daroma prielaida, kad vertinami dydžiai yra atsitiktiniai. Natūralu, kad, siekiant stebėti realių neatsitiktinių signalų dinamiką realiuoju laiku, tokie metodai nėra tinkami.

Kadangi širdies ir kraujagyslių ligos yra viena iš dažniausių mirties priežasčių, o kardiologijoje plačiausiai naudojamas neinvazinis tyrimas yra elektrokardiogramos registravimas, todėl labai svarbu mokėti laiku pastebėti menkiausius elektrokardiografinius pokyčius. Beveik prieš dešimtį metų lietuvių mokslininkai pasiūlė metodiką, skirtą signalų tarpusavio ryšio dinamikai nagrinėti. Pasiūlyta metodika, paremta matricų struktūrine analize, davė gerus rezultatus atliekant elektrokardiografinių signalų parametrų tyrimus ir buvo aprobuota daugybėje publikacijų ir projektų. Dėl minėtų priežasčių šiame darbe toliau plėtojama matricine analize paremtos elektrokardiografinių signalų tyrimo metodikos koncepcija. Ankstesniuose tyrimuose vertintos charakteristikos papildytos naujais matricos struktūrą nusakančiais įverčiais. Taip siekiama nustatyti, kaip dviejų signalų tarpusavio ryšį nusakančios kreivės forma (morfologija) ir kompleksiškumas susiję su fiziologiniais ir patologiniais procesais.

Struktūriniai matricų dėstiniai rado taikymo nišą ne tik kardiologijoje, bet ir iteraciniuose modeliuose. Skaliariniame iteraciniame modelyje kintamasis buvo pakeistas antros eilės kvadratine matrica. Taip buvo gauta modifikuota iteracinių modelių klasė, pasižyminti efektais, kurie nebūdingi skaliariniams modeliams ar jų plėtiniams. Buvo pastebėta, kad jeigu matricinio iteracinio modelio pradinių sąlygų matrica yra nulpotentinė, tuomet sistemos sprendiniai gali diverguoti. Esant idempotentinei matricai, toks elgesys neįmanomas. Matriciniai iteraciniai modeliai gali būti pritaikomi koduojant informaciją.

Nors aktualumą daugiausia lemia praktiniai taikymai, tačiau šio darbo pagrindas yra specifinės antros eilės matricų struktūros nusakymas. Remiantis matricų dėstiniais, buvo pasiūlyta signalų tarpusavio ryšius nagrinėjanti metodika, kuri pasižymi geromis savybėmis vertinant sportininkų fizinį pasirengimą ir atliekant klinikinius tyrimus. Matriciniai iteraciniai modeliai atskleidžia visiškai kitokias sistemos savybes nei nagrinėjami kitokie to paties iteracinio modelio plėtiniai. Minėti teiginiai nulemia šio disertacinio darbo aktualumą tiek teorine, tiek taikomąja prasme.

**Tyrimų objektas** – antros eilės matricų sekos, sugeneruotos matricinių iteracinių modelių bei matricų sekos, gautos sudarant jas iš elektrokardiogramos parametrų duomenų.

**Darbo tikslas** – išnagrinėti antros eilės matricų dėstinius (trijų, keturių komponenčių), ištirti matricinius iteracinius modelius ir išplėsti reikšmingų matricos struktūrą nusakančių įverčių aibę bei pritaikyti juos elektrokardiografinių signalų dinamikos stebėsenai.

**Suformuluotiems tikslams pasiekti darbe yra sprendžiami tokie uždaviniai:**

*Teoriniai:*

- Sudaryti idempotentų ir nulpotentų sistemos generavimo schemą. Įvesti idempotentų ir nulpotentų parametrines išraiškas.
- Sukonstruoti antros eilės matricų struktūrinius dėstinius idempotentais bei nulpotentais ir išnagrinėti dėstinių savybes.
- Išvesti būtinas ir pakankamas sąlygas matricų iteracinių modelių sprendiniams diverguoti.

*Praktiniai:*

- Tęsti ir plėtoti ankstesnes (KTU biomedicinos inžinerijos, LSMU, Kardiologijos instituto, Sporto instituto bei LSU mokslininkų) metodikas, paremtas antros eilės matricų struktūrine analize. Praplėsti matricos struktūrą nusakančių reikšmingų įverčių aibę ir pritaikyti elektrokardiografinių parametrų tarpusavio ryšio dinamikai tirti.

**Tyrimų metodai ir programinės priemonės:**

- Atliekant tyrimus plačiai naudojama matricų struktūrinės analizės teorija. Lyginamajai analizei atlikti panaudoti MATLAB paketo matematinės ir statistinės analizės metodai.
- Netiesinių dinaminių sistemų modeliai ir tyrimo metodai panaudoti matricų iteraciniams modeliams tirti.
- Matricų struktūrinė analizė pritaikyta elektrokardiografinių signalų sąsajų tyrimui atlikti.

**Darbo mokslinis naujumas ir praktinė svarba:**

- Disertacinio tyrimo metu buvo sudaryti *specialūs* antros eilės matricų dėstiniai iš nulpotentų ir idempotentų.
- Suformuota *modifikuota* matricinių iteracinių modelių klasė $\mathbf{X}^{(n+1)} = f(\mathbf{X}^{(n)})$, čia $\mathbf{X}^{(n)}$ yra antros eilės matrica, o funkcija $f$ – analizinė funkcija. Tokia modelių klasė atskleidžia savybes, kurios nėra būdingos nei vienmačiams iteraciniams modeliams ar jų plėtiniams, nei susietų iteracinių modelių tinkleliams.
- Dviejų elektrokardiografinių signalų tarpusavio ryšiui tirti pasiūlytos *naujos* charakteristikos, kurios leidžia pastebėti ir įvertinti kokybiškai naujus signalo pokyčius. Šios metodikos taikomos žmogaus sveikatos būklei stebėti ir vertinti.

**Ginti pateikiama:**

- *Specialūs* antros eilės matricų dėstiniai idempotentų bei nulpotentų tiesiniu dariniu.
- *Modifikuota* matricinių iteracinių modelių klasė $\mathbf{X}^{(n+1)} = f(\mathbf{X}^{(n)})$, kai $\mathbf{X}^{(n)} \in \mathbb{R}_{2\times 2}$, o $f$ – analizinė funkcija. Suformuluotos būtinos ir pakankamos sąlygos tokių iteracinių modelių sprendiniams diverguoti.
- *Nauji* įverčiai dviejų signalų tarpusavio ryšiui vertinti, paremti antros eilės matricų struktūrine analize. Šie įverčiai įgalina vertinti kokybiškai naujus elektrokardiografinio signalo pokyčius, palyginti su klasikine EKG analizės metodika.

**Darbo rezultatų aprobavimas:**

Darbo tema pateikti 12 moksliniai straipsniai, iš jų 5 mokslinės informacijos instituto duomenų bazės (ISI) leidiniuose, turinčiuose citavimo indeksą, dvi publikacijos atspausdintos tarptautinėse ir 3 nacionalinėse leidyklose. Likusios 7 publikacijos pristatytos kitų tarptautinių duomenų bazių leidiniuose. Disertacijos rezultatai buvo pristatyti 7 tarptautinėse konferencijose.

**Darbo apimtis ir struktūra:**

Daktaro disertaciją sudaro įvadas, 4 pagrindiniai skyriai, išvados, praktinė svarba, literatūros sąrašas ir publikacijų sąrašas. Disertacijos apimtis – 118 puslapių. Disertacijoje yra 40 paveikslų ir 208 šaltinių cituojamos literatūros aprašas.

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