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On Error Estimation and Convergence of the Difference Scheme for a Nonlinear Elliptic Equation with an Integral Boundary Condition

Regimantas Čiupaila [1,*], Mifodijus Sapagovas [2], Kristina Pupalaigė [3,4] and Gailė Kamilė Šaltenienė [1]

1 Department of Mathematical Modelling, Vilnius Gediminas Technical University, Saulėtekio al. 11, LT-10223 Vilnius, Lithuania; gaile.kamile.salteniene@vilniustech.lt
2 Institute of Data Science and Digital Technologies, Vilnius University, Akademijos g. 4, LT-08412 Vilnius, Lithuania; mifodijus.sapagovas@mif.vu.lt
3 Department of Applied Mathematics, Kaunas University of Technology, Studentų g. 50, LT-51368 Kaunas, Lithuania; kristina.pupalaige@ktu.lt
4 Department of Mechanical and Civil Engineering, Kaunas University of Applied Engineering Sciences, Tvirtovės al. 35, LT-50155 Kaunas, Lithuania
* Correspondence: regimantas.ciupaila@vilniustech.lt

Abstract: In this paper, a two-dimensional nonlinear elliptic equation with an integral boundary condition depending on two parameters is investigated. The problem is solved using the finite difference method. The error in the solution is evaluated based on the properties of M-matrices, and herewith the convergence of the difference scheme is proved. The majorant is constructed to estimate the error of the solution of the system of difference equations.

Keywords: nonlinear elliptic equation; nonlocal boundary condition; difference eigenvalue problem; M-matrix; construction of the majorant

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1. Introduction and Problem Statement

In this paper, we consider the boundary value problem for a nonlinear elliptic equation with a nonlocal integral condition that depends on two parameters, ξ and γ:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y,u), \quad (x,y) \in \Omega = \{0 < x < 1, 0 < y < 1\}, \tag{1}$$

$$u(x,0) = \mu_1(x), \quad u(x,1) = \mu_2(x), \quad u(0,y) = \mu_3(y), \tag{2}$$

$$u(1,y) = \gamma \int\limits_{\xi}^{1} u(x,y)dx + \mu_4(y), \tag{3}$$

where

$$\gamma \geq 0, \quad 0 \leq \xi < 1.$$

In recent decades, boundary value problems for differential equations with nonlocal conditions have become a rapidly developing field in the theory of differential equations and numerical analysis. The first two articles on the heat equation with integral boundary conditions [1,2] sparked significant interest among mathematicians, leading to extensive research in this area.

Mathematical models incorporating various types of nonclassical unconventional boundary conditions have been formulated to describe real-world processes and phenomena in fields such as thermoelasticity, population dynamics, inverse problems, control theory, biological processes, chemical diffusion, etc. [1,3–5] (see also survey papers [6–8]).

Additionally, boundary value problems with nonlocal conditions can be viewed as theoretical generalizations of classical boundary problems. For instance, [9] was written with this intention, anticipating further investigations and applications. Subsequent research by other authors has focused on specific nonlocal conditions and their practical applications.

Various numerical methods have been employed to solve boundary value problems for elliptic equations with nonlocal conditions, with the finite difference method being one of the most commonly used approaches. In [10], a difference scheme of second-order accuracy for an elliptic equation with a Bitsadze–Samarskii-type nonlocal integral condition was theoretically substantiated. The well-possessedness of this difference scheme in Hölder space is proved. A multi-dimensional elliptic equation with integral boundary conditions was examined in [11]. In [12], the Poisson equation in a rectangular domain with integral conditions was solved. It was proven that the corresponding difference scheme converges in the energy norm with a convergence rate of $O(h^{s-1})$, $1 < s \leq 3$, depending on the smoothness of the solution. The author of [13] presents iterative methods for the numerical solution of a class of nonlinear reaction-diffusion equations with nonlocal boundary conditions for both time-dependent and steady-state cases.

The eigenvalue problem, existence, and dynamic behaviour of solutions for linear and semi-linear elliptic and parabolic equations with nonlocal integral boundary conditions are investigated in [13]. A novel approach for solving the elliptic equation with two nonlocal conditions was introduced in [14].

Several papers have addressed high-order accuracy schemes. In [15], a fourth-order difference scheme for the two-dimensional Poisson equation in a rectangular domain with two integral conditions was constructed. The eigenvalue problem for this difference scheme is also considered. A difference scheme of fourth-order accuracy for the Laplace equation with a nonlocal Bitsadze–Samarskii condition was investigated in [16]. Another difference scheme of fourth-order accuracy for the Poisson equation with an integral condition was developed in [17], where it was proven that this scheme can achieve an asymptotic optimal error estimate in the maximum norm using the Fourier transformation.

A separate investigation problem in the finite difference method are the methods of solution in systems of difference equations. The matrices of these systems, which incorporate nonlocal conditions, exhibit two notable properties. First, in all cases except the ones with periodic conditions, the matrix is nonsymmetric. Second, the structure of spectrum of this matrix can change significantly depending on the values of varying parameters or functions in the nonlocal conditions. In recent years, increased attention has been given to two iterative methods: those based on the theory of M-matrices (the principle of regular splitting) [18,19] and ADI methods [20]. In analysing the convergence conditions for these iterative methods, many researchers have studied the structure of spectrum of difference equation systems with nonlocal conditions. As a result, investigations into nonlocal conditions have contributed to the study of eigenvalue problems for nonsymmetric matrices.

The structure of the spectrum of a one-dimensional eigenvalue problem with an integral nonlocal condition in the form (3) was examined in [21], while the structure of spectrum of the corresponding difference eigenvalue problem was analysed in [22,23]. Investigations into spectrum structures with other integral and two-point nonlocal conditions were conducted in [21,22,24–28]. The structure of the spectrum for an ordinary differential equation with nonlocal conditions (3) was investigated in [29].

Although relatively few studies focus on the practical applications of elliptic equations with nonlocal conditions, some notable examples exist. One such application involves the design of electrical contacts controlled by magnetic interaction forces [30], where the nonlinear equation of a surface with prescribed mean curvature serves as the mathematical model. Another example is a boundary value problem for the Vlasov–Poisson equation, which plays a role in designing thermonuclear fusion reactors [5].

Various numerical methods have been applied to solve elliptic equations with nonlocal conditions. The finite element method has been used for different forms of elliptic equations with various nonlocal conditions [31–34]. In [35], a composite scheme based on the finite element method is presented, with a proof of third-order convergence. In [36,37], the discrete comparison principle for the finite difference and finite elements methods is considered. Other approaches, including the Legendre–Galerkin spectral method and its modifications [38,39] and the radial basis function method [40], have also been employed. A comparative analysis of various numerical methods is provided in [41]. The linear and semi-linear elliptic equations with integral Neumann boundary conditions were investigated in [42,43].

In recent years, interest in new mathematical models has grown. One such model is fractional elliptic equations and their applications [44–46]. These studies have been motivated by both emerging practical problems and inside mathematical developments. Additionally, initial research on stochastic differential equations with nonlocal conditions has appeared [47].

In this paper, we consider the boundary value problem (1)–(3) for the nonlinear Poisson Equation (1) with the integral condition (3). Our investigation is based on the theory of M-matrices, with the primary goal of proving the convergence of the difference scheme. Specifically, we aim to determine values of $\gamma > 0$ for which the matrix of the difference equation system is an M-matrix. To prove the convergence of the difference scheme in the norm C, we construct a majorant. While our approach follows the standard method of proving convergence, we introduce a new idea: in the standard method, a majorant is typically constructed if the maximum principle is valid for the difference scheme. However, we demonstrate that the maximum principle is not a necessary condition for convergence; instead, it suffices for the matrix of the difference problem to be an M-matrix.

The values of the solution of a two-dimensional problem (1)–(3) in one coordinate direction are connected by an integral nonlocal condition (3). This is the characteristic approach of nonlocal conditions for elliptic equation in two- or multi-dimensional cases [10–12].

We have used this approach in our previous studies [18] as well, proving the convergence of iterative methods for both linear and nonlinear elliptic equations with nonlocal conditions. We use our previous results [23,48,49] from the application of M-matrices to the theoretical study of the system of difference equations, showing that the theory of M-matrices could be treated as an extension of the maximum principle in the case where the matrix of the system of difference equations is not diagonally dominant.

The structure of this paper is as follows: In Section 2, we construct the difference problem corresponding to the differential problem (1)–(3). In Section 3 we present some results from M-matrix theory. Section 4 investigates the difference eigenvalue problem. Based on the spectral structure of the difference problem, Section 5 presents a comparison theorem. There, we construct the majorant for the error in the solution. This error is estimated in Section 6. Numerical results illustrating the details of the error estimates are provided in Section 7. Finally, Section 8 presents conclusions and possible generalizations.

2. Difference Problem

Suppose that the differential problem (1)–(3) has a unique, sufficiently smooth solution such that its derivatives up to the fourth order are bounded. Then, we write the following difference problem:

$$\frac{U_{i-1,j}-2U_{ij}+U_{i+1,j}}{h^2}+\frac{U_{i,j-1}-2U_{ij}+U_{i,j+1}}{h^2}=f(x_i,y_j,U_{ij}), \quad i,j=1,2,\dots,N-1, \tag{4}$$

$$U_{i0}=(\mu_1)_i,\ U_{iN}=(\mu_2)_i,\ U_{0j}=(\mu_3)_j,\ i,j=0,1,\dots,N, \tag{5}$$

$$U_{Nj}=h\gamma\left(\frac{U_{mj}+U_{Nj}}{2}+\sum_{i=m+1}^{N-1}U_{ij}\right)+(\mu_4)_j,\ j=1,2,\dots,N-1, \tag{6}$$

where

$$h=\frac{1}{N}, \quad \xi=mh$$

and N, m arenatural numbers.

We denote u_{ij} as the solution of the differential problem (1)–(3) and U_{ij} as the solution of difference problem (4)–(6). For the solution u_{ij} of differential problem (1)–(3), the system of difference equations is as follows:

$$\frac{u_{i-1,j}-2u_{ij}+u_{i+1,j}}{h^2}+\frac{u_{i,j-1}-2u_{ij}+u_{i,j+1}}{h^2}=f(x_i,y_j,u_{ij})+R_{ij}(h), \qquad i,j=1,2,\dots,N-1, \tag{7}$$

$$u_{i0}=(\mu_1)_i,\ u_{iN}=(\mu_2)_i,\ u_{0j}=(\mu_3)_j,\ i=0,1,\dots,N,\ j=0,1,\dots,N, \tag{8}$$

$$u_{Nj}=h\gamma\left(\frac{u_{mj}+u_{Nj}}{2}+\sum_{i=m+1}^{N-1}u_{ij}\right)+(\mu_4)_j+R_{Nj}(h),\ j=1,2,\dots,N-1, \tag{9}$$

where N, m are natural numbers.

The following approximations are correct for approximation errors (under the condition that the solution of the differential problem (1)–(3) is sufficiently smooth):

$$|R_{ij}(h)|\le\frac{h^2}{6}M_4, \quad i,j=1,2,\dots,N-1, \tag{10}$$

$$|R_{Nj}(h)|\le\frac{h^2}{12}\gamma M_2, \quad j=0,1,\dots,N-1, \tag{11}$$

where

$$M_2=max\left(\left|\frac{\partial^2 u}{\partial x^2}\right|\right), \qquad M_4=max\left(\left|\frac{\partial^4 u}{\partial x^4}\right|,\left|\frac{\partial^4 u}{\partial y^4}\right|\right).$$

Then, the error could be defined as

$$z_{ij}=u_{ij}-U_{ij}. \tag{12}$$

From Equations (4)–(6) and Equations (7)–(9), we obtain the following system for the error

$$\frac{z_{i-1,j}-2z_{ij}+z_{i+1,j}}{h^2}+\frac{z_{i,j-1}-2z_{ij}+z_{i,j+1}}{h^2}=\frac{\partial\tilde{f}}{\partial u}z_{ij}+R_{ij}(h), \quad i,j=1,2,\dots,N-1, \tag{13}$$

$$z_{i0}=z_{iN}=z_{0j}=0, \quad i=0,1,\dots,N, \quad j=0,1,\dots,N, \tag{14}$$

$$z_{Nj}=h\gamma\left(\frac{z_{mj}+z_{Nj}}{2}+\sum_{i=m+1}^{N-1}z_{ij}\right)+R_{Nj}(h), \quad j=1,2,\dots,N-1. \tag{15}$$

From the nonlocal condition (15), we express

$$z_{Nj} = \sum_{i=1}^{N-1} \alpha_i z_{ij} + \beta R_{Nj}(h), \tag{16}$$

where

$$\alpha_i = \begin{cases} 0, & i = 1,2,\ldots,m-1, \\ \dfrac{h\gamma}{2-h\gamma}, & i = m, \\ \dfrac{2h\gamma}{2-h\gamma}, & i = m+1, m+2, \ldots, N-1, \end{cases}$$

and

$$\beta = \frac{2}{2-h\gamma}, \quad m = \frac{\xi}{h}.$$

We enter z_{Nj} from (16) into Equation (13), where $i = N-1$. Then, we rewrite the system (13)–(15) in another form

$$\frac{z_{i-1,j} - 2z_{ij} + z_{i+1,j}}{h^2} + \frac{z_{i,j-1} - 2z_{ij} + z_{i,j+1}}{h^2} = \frac{\partial \tilde{f}}{\partial u} z_{ij} + R_{ij}(h),\ i = 1,2,...,N-2,\ j = 1,2,...,N-1, \tag{17}$$

$$\frac{z_{N-2,j} - 2z_{N-1,j} + \sum\limits_{i=1}^{N-1} \alpha_{ij} z_{ij}}{h^2} + \frac{z_{N-1,j-1} - 2z_{N-1,j} + z_{N-1,j+1}}{h^2} = \frac{\partial \tilde{f}}{\partial u} z_{Nj} + R_{Nj}(h),\ j = 1,2,\ldots,N-1, \tag{18}$$

$$z_{i0} = z_{iN} = z_{0j} = 0, \quad i = 0,1,\ldots,N, \quad j = 0,1,\ldots,N. \tag{19}$$

Remark 1. *The system (13)–(15) is equivalent to the system (17)–(19) along with (16). It is important to note that by rearranging the system (13)–(15), we separated the system (17)–(19) without the nonlocal condition. The system (17)–(19), which contains $(N-1)^2$ equations and $(N-1)^2$ unknowns z_{ij}, $i,j = 1,2,\ldots,N-1$, can be investigated separately (without nonlocal conditions).*

The system (17)–(19) can be presented in matrix form

$$Az + Dz = r(h). \tag{20}$$

Here, $r(h) = \{r_{ij}(h)\}$ is vector

$$r_{ij}(h) = \begin{cases} R_{ij}(h) & i = 1,2,\ldots,N-1;\ j = 1,2,\ldots,N-1, \\ R_{N-1,j} + \dfrac{\beta}{h^2} R_{N,j} & i = N-1;\ j = 1,2,\ldots,N-1. \end{cases}$$

Matrix D in (20) is the diagonal matrix obtained from the right side of the difference problem. Matrix A is defined as follows:

$$A = h^{-2}(\Lambda_x + C + \Lambda_y),$$

where Λ_x and Λ_y are block matrices corresponding to the second-order partial derivatives with the respect to variables x and y, and C is the block matix obtained from the following nonlocal condition:

$$\Lambda_x = \begin{pmatrix} \Lambda_o & & & \\ & \Lambda_o & & \\ & & \ddots & \\ & & & \Lambda_o \end{pmatrix}, \quad \Lambda_y = \begin{pmatrix} 2I & -I & & \\ -I & 2I & -I & \\ & \ddots & \ddots & \ddots \\ & & -I & 2I \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & & & \\ & C_1 & & \\ & & \ddots & \\ & & & C_1 \end{pmatrix}$$

and

$$\Lambda_o = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & . & . & . \\ -\alpha_1 & -\alpha_2 & \dots & -\alpha_{N-1} \end{pmatrix}.$$

Matrices I, Λ_0 and C_1 are of the order $N-1$. Matrices Λ_x, Λ_y, C, and A are of the order $(N-1)^2$.

3. Application of M-Matrices

When we write the system in matrix form (20), the elements of matrix A satisfy the following conditions: diagonal elements $a_{ii} > 0$, non-diagonal elements $a_{ij} \leq 0$, $i \neq j$. Therefore, we will need the notion of an M-matrix for further examination of the matrix A.

Definition 1 ([50,51])**.** *A real square matrix $B = \{b_{kl}\}$, $k, l = 1, 2, \dots, n$, with $b_{kl} \leq 0$ for all $k \neq l$ is called an M-matrix if B^{-1} is non-singular and all elements of B^{-1} are non-negative.*

If the condition $b_{kl} \leq 0$ is fulfilled, the following two conditions are equivalent:

(1) B^{-1} exists and $B^{-1} \geq 0$;

(2) The real parts of all eigenvalues of the matrix B are positive.

If $h\gamma < 1$, then diagonal elements of matrix A from (20) are positive. Non-diagonal elements of matrix A are always non-positive.

We will estimate the error in the solution of the system (4)–(6). We will show that matrix A is an M-matrix.

For the error

$$z_{ij} = u_{ij} - U_{ij}$$

matrix form of the system of difference equations is

$$Az + Dz = r(h).$$

Here, D is the diagonal matrix : $D = \{d_{ij}\}$, $d_{ij} \geq 0$, $|r_{ij}(h)| \leq R_{ij}(h)$.

4. Difference Eigenvalue Problem

The eigenvalue problem for matrix A is equivalent to the two-dimensional difference eigenvalue problem [23]:

$$\frac{u_{i-1,j} - 2u_{ij} + u_{i+1,j}}{h^2} + \frac{u_{i-1,j} - 2u_{ij} + u_{i,j+1}}{h^2} + \lambda u_{ij} = 0, \quad i, j = 1, 2, \dots, N-1, \tag{21}$$

$$u_{i0} = 0, \quad u_{iN} = 0, \quad u_{0j} = 0, \quad i, j = 0, 1, \dots, N \tag{22}$$

$$u_{Nj} = h\gamma \left(\frac{u_{mj} + u_{Nj}}{2} + \sum_{i=m+1}^{N-1} u_{ij} \right), \quad j = 1, 2, \dots, N-1. \tag{23}$$

Using the Fourier method, we separate variables in problem (21)–(23) as [23]

$$u_{ij} = v_i \cdot w_j.$$

In this way, the two-dimensional eigenvalue problem (21)–(23) is reduced to two one-dimensional problems

$$\frac{v_{i-1} - 2v_i + v_{i+1}}{h^2} + \eta v_i = 0, \quad i = 1, 2, \ldots, N-1, \tag{24}$$

$$v_0 = 0, \quad v_N = h\gamma\left(\frac{v_m + v_N}{2} + \sum_{i=m+1}^{N-1} v_i\right) \tag{25}$$

and

$$\frac{w_{j-1} - 2w_j + w_{j+1}}{h^2} + \mu w_j = 0, \quad j = 1, 2, \ldots, N-1, \tag{26}$$

$$w_0 = 0, \quad w_N = 0. \tag{27}$$

To prove that matrix A is an M-matrix, we will examine the eigenvalues of the matrix of the problem (24)–(25).

It is shown in [21] that

- if γ and ξ are real, then eigenvalues of matrix A are real;
- if $\gamma < \dfrac{2}{1-\xi^2}$, then all eigenvalues ($\lambda > 0$) are positive;
- there exists a negative eigenvalue if and only if $\gamma > \dfrac{2}{1-\xi^2}$.

The eigenvalues of the problem (26), (27) are

$$\mu_l = \frac{4}{h^2}\sin^2\left(\frac{l\pi k}{2}\right), \quad l = 1, 2, \ldots, N-1. \tag{28}$$

Now

$$\lambda_{kl} = \eta_k + \mu_l, \quad k, l = 1, 2, \ldots, N-1. \tag{29}$$

Suppose, the parameters ξ and γ of the eigenvalue problem (24), (25) satisfy the condition

$$\gamma < \frac{2}{1-\xi^2}. \tag{30}$$

It is proved in [22,23] that, under condition

$$\gamma < \frac{2}{1-\xi^2}$$

all the eigenvalues $\eta_k > 0$, $k = 1, 2, \ldots, N-1$. If $\gamma = \dfrac{2}{1-\xi^2}$, then $\eta_1 = 0$, $\eta_k > 0$, $k = 2, \ldots, N-1$.

We will formulate the statement on eigenvalues η_k when parameters ξ and γ satisfy condition (30).

Conclusion 1: [22,27] One negative eigenvalue

$$\eta_1 = -\frac{4}{h^2}\sinh^2\left(\frac{\beta_0 h}{2}\right) \tag{31}$$

of eigenvalue problem (24), (25) exists if and only if the parameters ξ and γ satisfy condition (30). Here, $\beta_0 > 0$ is unique positive root of the equation

$$\sinh(\beta) = \frac{\gamma h}{2}\frac{\cosh(\beta) - \cosh(\beta\xi)}{\tanh\left(\dfrac{\beta h}{2}\right)}. \tag{32}$$

The rest of $\eta_k, k = 2, \ldots, N-1$ are positive.

We will need another interpretation of the expression (32). With that aim, we write down (32) in another form:

$$\gamma = \frac{2}{h}\tanh\left(\frac{\beta h}{2}\right)\frac{\sinh(\beta)}{\cosh(\beta) - cosh(\beta\xi)}. \tag{33}$$

Conclusion 2: Suppose the value of the parameter ξ in the eigenvalue problem (24), (25) is fixed, $\xi_0 \in [0,1)$ and the parameter γ is not known. It should be chosen such that the negative eigenvalue η_1, as provided by formula (31), should exist for the given value of β_0. According to Conclusion 1, the parameter γ_1 should be chosen from formula (33)

$$\gamma_1 = \frac{2}{h}\tanh\left(\frac{\beta_0 h}{2}\right)\frac{\sinh(\beta_0)}{\cosh(\beta_0) - \cosh(\beta_0\xi_0)}.$$

Conclusion 2 will be directly used to prove the following theorem.

In [23], it was proved (see propositions 8 and 9) that the value β_0, which describes the negative eigenvalue η_1 of the problem (24), (25) exhibits certain monotonicity properties. We will now formulate these properties, paraphrasing some of them.

Conclusion 3: Suppose that along the concrete values of $\xi_0 \in [0,1)$ and $\gamma_1 > 0$, for which the condition (30) is true, the root of Equation (32) is $\beta_0 > 0$. If we increase ξ_0, i.e., we take $\overline{\xi_0} > \xi_0$ such that for parameters $\overline{\xi_0}$, γ_1 conditions (30) are true; then, the root β_0 of Equation (32) will decrease: $\overline{\beta_0} < \beta_0$. Analogously, if the value of γ_1 decreases ($\overline{\gamma_1} < \gamma_1$) in such a way that condition (30) is true for the parameters ξ_0 and γ_1, then the root of Equation (32) will decrease.

Now we can formulate and prove one of the main results of the paper.

Theorem 1. *For any value of $\xi_0 \in [0,1)$, there exists $\gamma_1 > 0$ such that for all values of $\xi \in [\xi_0, 1)$ and $\gamma \in [0, \gamma_1)$, the eigenvalues of the two-dimensional difference problem (21)–(23) (the eigenvalues of the matrix A) are non-negative. The eigenvalue $\lambda_{11} = 0$ exists only in the case where $\xi = \xi_0$, $\gamma = \gamma_1$.*

Proof. We consider the eigenvalue problem (24), (25) with the specific values $\xi^{(i)}$ and $\gamma^{(i)}$ for $i = 1,2,3$, of the parameters ξ and γ. For every pair of parameters $\xi^{(i)}$ and $\gamma^{(i)}$, $i = 1,2,3$, we find $\beta_0^{(i)}$, $\eta_1^{(i)}$ and $\lambda_{11}^{(i)}$.

Values $\xi^{(1)}$ and $\gamma^{(1)}$ will be defined as follows. We choose any $\xi_0 \in [0,1)$ and define the value β_0 by the equality

$$|\eta_1| = \mu_1, \tag{34}$$

where η_1 and μ_1 are the least eigenvalues of the problems (24), (25) and, respectively, (26), (27). It follows from (34)

$$\frac{4}{h^2}\sinh^2\left(\frac{\beta_0 h}{2}\right) = \frac{4}{h^2}\sin^2\left(\frac{\pi h}{2}\right).$$

From here, we obtain

$$\beta_0 = \frac{2}{h}\ln\left(\sin^2\left(\frac{\pi h}{2}\right) + \sqrt{\sin^2\left(\frac{\pi h}{2}\right) + 1}\right). \tag{35}$$

Now we define γ_1 in such a way that the problem (24), (25) possesses a negative eigenvalue η_1, defined by formula (31), where β_0 is a value found from (35). According to Conclusion 2, the value of γ_1 should be defined as

$$\gamma_1 = \frac{2}{h}\tanh\left(\frac{\beta_0 h}{2}\right)\frac{\sinh(\beta_0)}{\cosh(\beta_0) - \cosh(\beta_0\xi_0)}. \tag{36}$$

Case 1. $\xi^{(1)} = \xi_0, \quad \gamma^{(1)} = \gamma_1.$

It follows from the definition of the parameters $\xi^{(1)}$ and $\gamma^{(1)}$ that the negative eigenvalue (31) $\eta_1^{(1)}$ of the problem (24), (25), with the values of parameters $\xi^{(1)} = \xi_0$ and $\gamma^{(1)} = \gamma_1$, exists. Here, β_0 is given by formula (35).

From here, we obtain

$$\lambda_{11}^{(1)} = \eta_1^{(1)} + \mu_1,$$
$$\lambda_{kl}^{(1)} \geq 0, \quad k,l = 1,2,\ldots,N-1.$$

Thus, $\lambda_{11} = 0$, $\lambda_{kl} \geq 0$, $k,l = 1,2,\ldots,N-1$, when $\xi = \xi_0$, $\gamma = \gamma_1$.

Case 2. We take $\xi^{(2)}$ as any value from the interval $(\xi^{(1)}, 1)$:

$$\xi^{(2)} \in (\xi^{(1)}, 1), \quad \gamma^{(2)} = \gamma_1.$$

Choosing the pair of the parameters for the problem (24), (25) in such way there are two possible variants:

(a) condition (30) is not fulfilled for the problem (24), (25) with the values of parameters $\xi^{(2)}$, $\gamma^{(2)}$. In other words, if negative value $\eta_1 < 0$ does not exist, then $\eta_1^{(2)} > 0$ and $\lambda_{11}^{(2)} = \eta_1^{(2)} + \mu_1 > 0$;

(b) condition (30) is true with the values $\xi^{(2)}$, $\gamma^{(2)}$, i.e., $\gamma^{(2)} > \dfrac{2}{1 - \left(\xi^{(2)}\right)^2}$.

As

$$\xi^{(2)} > \xi^{(1)}, \quad \gamma^{(2)} = \gamma^{(1)},$$

then, according to Condition 3,

$$\beta_0^{(1)} < \beta_0^{(2)}.$$

So

$$|\eta_1^2| < |\eta_1^{(1)}|,$$
$$\lambda_{11}^{(2)} = \eta_1^{(2)} + \mu_1 > \eta_1^{(1)} + \mu_1 > 0.$$

Thus, $\lambda_{kl} > 0$, $kl = 1,2,\ldots,N-1$, when $\xi \in (\xi_0, 1)$, $\gamma = \gamma_1$.

Case 3. Now we define

$$\xi^{(3)} = \xi^{(2)} \in (\xi_0, 1), \quad \gamma^{(3)} \in (0, \gamma^{(2)}).$$

Analogously, as it was for $\xi^{(2)}$ and $\gamma^{(2)}$, there are two variants:

(a) if condition (30) with $\xi^{(3)}$, $\gamma^{(3)}$ is not true, then there is no $\eta^{(3)} < 0$;

(b) if condition (30) with $\xi^{(3)}$, $\gamma^{(3)}$ is true, then

$$\gamma^{(3)} > \frac{2}{1 - \left(\xi^{(2)}\right)^2}.$$

Analogously, as
$$\gamma^{(3)} < \gamma^{(2)}, \quad \xi^{(3)} = \xi^{(2)},$$
then
$$\beta_0^{(3)} < \beta_0^{(2)} < \beta_0,$$
$$|\eta_1^{(3)}| < |\eta_1^{(2)}| < |\eta_1|$$
and
$$\lambda_{11}^{(3)} = \eta_1^{(3)} + \mu_1 > 0.$$

Thus, $\lambda_{kl} > 0, k, l = 1, 2, \ldots, N-1$ when $\xi \in (\xi_0, 1)$, $\gamma \in (0, \gamma_1)$.

The Theorem is proved. □

Corollary 1. *If $0 \leq \gamma \leq \gamma_1$, then the matrix A is an M-matrix.*

Indeed, as stated earlier, $a_{kl} \leq 0$, $k \neq l$, $a_{kk} > 0$ and according to Theorem 1 and Definition 1, $\lambda(A) > 0$, A is an M-matrix.

Corollary 2. *The theorem can be reformulated based on the information from the graph. If the point $(\overline{\xi}, \overline{\gamma})$ lies under the curve (Figure 1), then the matrix A, with $\xi = \overline{\xi}$, $\gamma = \overline{\gamma}$ will be an M-matrix.*

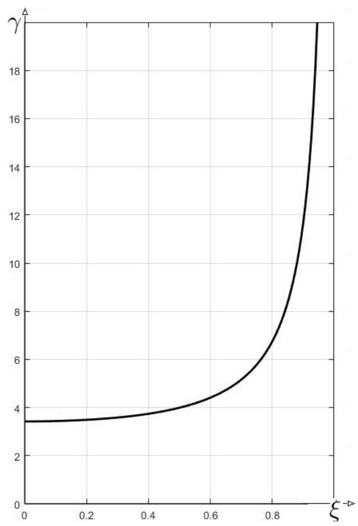

Figure 1. Relation graph of γ depending of ξ (formula (33)).

We also showed that in (20), both matrix A and matrix $(A + D)$ are M-matrices for $\xi \in [0, 1)$ and $\gamma \in [0, \gamma_1)$.

5. Comparison Theorem and Construction of the Majorant Function

Our main aim is to estimate the error occurring from the nonlocal condition. In [49], based on the properties of M-matrices, a statement was proved, which is commonly referred to as the comparison theorem in the theory of finite difference method. According to this theorem, a majorant function can be formed and the error is estimated using this function.

Lemma 1 ([49]). *Suppose that u and w are the solutions of two systems*
$$(A + D)u = f \tag{37}$$
$$Aw = g, \tag{38}$$

respectively, where A is an M-matrix, $D \geq 0$ is diagonal matrix, and $g \geq 0$. If $|f| \leq g$, then $|u| \leq w$.

The solution w of the system (38) is called the majorant of the solution to the system (37).

This method for estimating the error in the solution using the majorant was introduced in 1930 by S. Gerschgorin [52]. In this paper, the Dirichlet boundary value problem for the two-dimensional Laplace equation was solved by the finite difference method.

The method, along with comments on its advantages and disadvantages, is described in detail in [53,54]. We would like to draw attention to one important aspect of the method: The comparison theorem has always been presented as a conclusion of the maximum principle [53–55]. The maximum principle is formulated for a system of difference equations whose matrix has the following properties:

- $a_{ij} \leq 0, i \neq j$;
- $a_{ii} > 0$;
- $a_{ii} \geq \sum |a_{ij}|$ (diagonally dominant).

It is a strong limitation. M-matrices do not require a diagonal dominance condition.

On the other hand, we must note that the result of Lemma 1, expressed in another form, essentially coincides with some results known in linear algebra (see, e.g., the discrete comparison principle in [56]). Following [56], we formulate this principle for a more general class of matrices than M-matrices.

Lemma 2 (discrete comparison principle [56])**.** *Let A be a monotone matrix $w \geq 0$. If $Av \leq Aw$ for $v, w \in R^n$, then it follows that $v \leq w$.*

However, to the best of our knowledge, this principle has not actually been applied to estimate the error in the approximate solution obtained by the difference methods (except in [18,48,49]).

The expression of the majorant function is usually not commented on too much. One disadvantage of using this method to estimate the error by the construction of the majorant is that, except in some simple cases, it is not clear how to construct the majorant itself.

We construct the majorant of the solution for the system (20) in the following form:

$$w(x,y) = \frac{h^2 M(1+\gamma)}{24}\left(K - \left(x - \frac{1}{2}\right)^2 - \left(y - \frac{1}{2}\right)^2 + ax\right), \tag{39}$$

where $K \geq \frac{3}{2}$, $a \geq 0$ are constants, whose values, in the general case, depend on ξ and γ. $M = max(M_2, M_4)$.

We note that under conditions $K \geq \frac{3}{2}$, $a \geq 0$, there will always be

$$w(x_i, y_j) \geq 0, \quad i,j = 0,1,\ldots,N. \tag{40}$$

We investigate for which values of ξ and γ function (39) $w(x,y)$ will be the majorant of the solution z_{ij} of the system (20). For this, we must write the analogue of Equations (17)–(19) for the function $w(x,y)$.

Thus, knowing the matrix A and the vector $w = \{w_{ij}\}$, we construct the system $Aw = g$, i.e., we estimate the coordinates of the vector $g = \{g_{ij}\}$, where $i,j = 1,2,\ldots,N-1$.

First, we write down the system of equations in the form (17)–(19) for the solution $w(x,y)$. It follows from (39) that

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial y^2} = -\frac{h^2 M(1+\gamma)}{12},$$
$$-\frac{w_{i+1,j} - 2w_{ij} + w_{i-1,j}}{h^2} - \frac{w_{i,j+1} - 2w_{ij} + w_{i,j-1}}{h^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}.$$

From here, we obtain the analogous equation to Equation (17) for the function $w(x,y)$.

$$-\frac{w_{i+1,j} - 2w_{ij} + w_{i-1,j}}{h^2} - \frac{w_{i,j+1} - 2w_{ij} + w_{i,j-1}}{h^2} = \frac{h^2 M(1+\gamma)}{6}, \quad i,j = 0,1,2,\dots,N-1. \tag{41}$$

Now, we will derive a simple sufficient condition under which the function $w(x,y)$, defined by formula (29), will be the majorant of z_{ij}. For this, we denote

$$g_{Nj}(\xi,\gamma) = w_{Nj} - h\gamma\left(\frac{w_{mj} + w_{Nj}}{2} + \sum_{i=m+1}^{N-1} w_{ij}\right), \quad j = 1,2,\dots,N-1, \tag{42}$$

where $\xi = mh$, $1 = Nh$.

The equivalent of conditions (19) for the function $w(x,y)$ is

$$w_{i0} = w(x_i,0), \quad w_{iN} = w(x_i,1), \quad w_{0j} = w(0,y_j). \tag{43}$$

Further, the difference equation system (41)–(43) for the function $w(x,y)$ is written in the form $Aw = g$. To achieve this, we express from Equation (42)

$$w_{Nj} = \sum_{i=1}^{N-1} \alpha_i w_{ij} + \beta g_{Nj}(\xi,\gamma), \quad j = 1,2,\dots,N-1, \tag{44}$$

where α_i and β are the same as in formula (16). After inserting these w_{Nj} expressions into Equations (41), when $i = N-1$, the system (41)–(44) is transformed into the form

$$-\frac{w_{i+1,j} - 2w_{ij} + w_{i-1,j}}{h^2} - \frac{w_{i,j+1} - 2w_{ij} + w_{i,j-1}}{h^2} = \frac{h^2 M(1+\gamma)}{6}, \; i = 1,2,\dots,N-1, \tag{45}$$

$$-\frac{w_{N-2,j} - 2w_{N-1,j} + \sum\limits_{i=1}^{N-1} \alpha_i w_{ij}}{h^2} - \frac{w_{N-1,j+1} - 2w_{N-1,j} + w_{N-1,j-1}}{h^2} = \frac{h^2 M(1+\gamma)}{6} + \frac{\beta}{h^2} g_{Nj}(\xi,\gamma), \tag{46}$$

where $j = 1,2,\dots,N-1$.

Now, the system (45), (46), (43) can be written down in matrix form

$$Aw = g \tag{47}$$

analogously to how system (20) was written. Matrix A in systems (20) and (47) is the same matrix.

We should admit that to obtain the coordinate expressions of vector $g = \{g_{ij}\}$, $i,j = 1,2,\dots,N-1$, the respective values of the solution w_{ij} of system (47) w_{ij}, w_{i0}, w_{iN} on the contour points are not equal to zero as was characteristic for solution z_{ij} of the system (20).

Therefore, the values w_{ij}, w_{i0}, w_{iN} should be incorporated into the expressions for g_{ij}. Then, we obtain

$$g_{ij} = \begin{cases} \dfrac{h^2M(1+\gamma)}{6} + \dfrac{\tilde{w_{ij}}}{h^2} \geq \dfrac{h^2M(1+\gamma)}{6}, & i = 1,2,\ldots,N-2, \\ \dfrac{h^2M(1+\gamma)}{6} + \dfrac{\beta}{h^2}g_{Nj} + \dfrac{\tilde{w_{ij}}}{h^2} \geq \dfrac{h^2M(1+\gamma)}{6} + \dfrac{\beta}{h^2}g_{Nj}, & i = N-1, \end{cases} \tag{48}$$

where $\tilde{w_{ij}}$ is one of the values $w_{ij} \geq 0$, $w_{i0} \geq 0$, and $w_{iN} \geq 0$ or is equal to zero.

We have already formulated two problems mentioned in Lemma 1. The first is the problem (20), i. e. $Az + Dz = r(h)$, with the expressions for $r(h)$

$$r_{ij}(h) = \begin{cases} R_{ij}(h), & i = 1,2,\ldots,N-2,\ j = 1,2,\ldots,N-1, \\ R_{N-1,j}(h) + \dfrac{\beta}{h^2}R_{Nj}, & i = N-1,\ j = 1,2,\ldots,N-1, \end{cases} \tag{49}$$

and estimates (10), (11) for r_{ij}. The second problem is system (47), in which g_{ij} are defined by formula (48). With regard to the expressions of r_{ij} and g_{ij} in (20) and (47), we formulate the conclusion from Lemma 1.

Conclusion 4: If matrix A is an M-matrix, then function $w(x,y)$, defined by formula (39), is the majorant of system (20) when the condition

$$g_{Nj} \geq \frac{h^2M(1+\gamma)}{6} \tag{50}$$

is satisfied.

Indeed, for $w(x,y)$ to be the majorant of system (20), the following inequalities should be correct:

$$|r_{ij}(h)| \leq g_{ij}, \quad i,j = 1,2,\ldots,N-1. \tag{51}$$

From estimations (10) and (11), it follows that if $i = 1,2,\ldots,N-2$, then

$$|r_{ij}(h)| = |R_{ij}(h)| \leq \frac{h^2M_4}{6} \leq \frac{h^2M(1+\gamma)}{6} \leq g_{ij}$$

Further, when $i = N-1$, and from definitions (48) and (49) for $r_{N-1,j}(h)$ and $g_{N-1,j}$, it follows that

$$|r_{N-1,j}(h)| \leq (1 + \frac{\beta}{h^2})\frac{h^2M(1+\gamma)}{6} \leq g_{N-1,j}$$

if only condition

$$|r_{N-1,j}(h)| \leq g_{N-1,j}$$

is satisfied.

According to Conclusion 4, we have a constructive algorithm to verify, for concrete values of ξ and γ, whether the function $w(x,y)$, defined by formula (39) is the majorant of the solution z_{ij}. On the other hand, by varying K and a, we obtain different majorants.

6. Evaluation of the Error

With reference to Lemma 1 and Conclusion 4, it is possible to evaluate the solution z_{ij} of Equation (20), i.e., to evaluate the error in the approximate solution obtained by the finite difference method. To achieve this, we list some comments related to the evaluation of the error.

Firstly, it is convenient to modify inequality (50) as the sufficient condition under which $w(x,y)$ is the majorant of the system (20).

Denote

$$w^*(x,y) = \frac{1}{4}\left(K\left(x-\frac{1}{2}\right)^2 - \left(y-\frac{1}{2}\right)^2 + ax\right) \tag{52}$$

and

$$g^*_{Nj} = w^*_{Nj} - \gamma h\left(\frac{w^*_{mj}+w^*_{Nj}}{2} + \sum_{i=m+1}^{N-1} w^*_{ij}\right). \tag{53}$$

Then

$$w(x,y) = \frac{h^2M(1+\gamma)}{6}w^*(x,y), \qquad g_{Nj} = \frac{h^2M(1+\gamma)}{6}g^*_{Nj}$$

and Conclusion 4 could be reformulated as follows:

Conclusion 5: For the function $w(x,y)$, defined by formula (39), to be a majorant of system (20) with the M-matrix, the sufficient condition is

$$g^*_{Nj} \geq 1, \tag{54}$$

where g^*_{Nj} is defined by formulas (52) and (53).

Secondly, we note that there are two variable parameters, ξ and γ, in the differential problem. This complicates providing a simple and exhaustive answer to the question of which values of the parameters ξ and γ make the function $w(x,y)$, defined by formula (39), a majorant of the solution z_{ij}, and how to optimally choose the coefficients K and a of majorant $w(x,y)$ depending on the values of ξ and γ—the authors, at least, have no consummate answer. However, by analysing the function $w(x,y)$ as a majorant, some results could be provided. First of all, we note that the function $g^*_{Nj}(\xi,\gamma)$ (as well as $g_{Nj}(\xi,\gamma)$) is monotonic with respect to both arguments.

1. Function $g^*_{Nj}(\xi,\gamma)$ is a monotonically increasing function with respect to the variable ξ in the interval $\xi \in [0,1)$.
2. Function $g^*_{Nj}(\xi,\gamma)$ is a monotonically decreasing function with respect to the variable γ in the interval $\gamma \in [0,\infty)$.

From this monotonicity, the conclusion follows.

Conclusion 6: Provided matrix A is an M-matrix when $\xi \in [0,1)$ and $\gamma \in [0,\gamma_1)$ (see formula (36)). Suppose, choosing concrete $\tilde{\xi} \in [0,1)$ and $\tilde{\gamma} \in [0,\gamma_1)$, we obtain

$$g^*_{Nj} \geq 1.$$

This means that $w(x,y)$ is a majorant and the estimation of the error is as follows:

$$|z_{ij}| \leq \frac{h^2M(1+\gamma)}{6}max|w^*(x,y)| \leq \frac{h^2M}{24}(1+\gamma)(K+a) = \frac{MC}{24}h^2, \tag{55}$$

where $C = (1+\gamma)(K+a)$.

The property follows from the monotonicity that

$$g^*(\xi,\gamma) \geq 1$$

is true not only in the point $(\tilde{\xi},\tilde{\gamma})$, but also in the area $S = \{\tilde{\xi} \leq \xi < 1,\ 0 \leq \gamma < \tilde{\gamma}\}$.

7. Numerical Results

We have performed a numerical experiment in order to supplement and clarify the theoretical statements on the construction of the majorant $w(x,y)$ defined by formula (39). We will use the following definitions and notations:

$$S_1 = \{\xi \in [0,1),\ \gamma \in [0,\gamma_1)\},$$
$$\gamma_1 = f(\beta_0, \xi_0) = \frac{2}{h}\tanh\left(\frac{\beta_0 h}{2}\right)\frac{\sinh(\beta_0)}{\cosh(\beta_0) - \cosh(\beta_0\xi_0)}, \quad (56)$$
$$\beta_0 = \frac{2}{h} ln\left(\sin\frac{\pi h}{2} + \sqrt{\sin^2\frac{\pi h}{2} + 1}\right).$$

Specifically, it was proved that, in the area S_1, all the eigenvalues of matrix A are real and positive:

- when $\gamma = \gamma_1$, then $\exists\lambda(A) = 0$;
- when $\gamma > \gamma_1$, then $\exists\lambda(A) < 0$.

In the numerical experiment, we will use a slightly stricter condition $g^*_{Nj} \geq 1$. Since

$$h\left(\frac{w_{mj} + w_{Nj}}{2} + \sum_{i=m+1}^{N-1} w_{ij}\right) \leq \int_{\xi}^{1} w(x)dx, \quad \text{when } \frac{\partial^2 u}{\partial x^2} \leq 0,$$

then

$$g^*_{Nj} = w^*_{Nj} - \gamma h\left(\frac{w^*_{mj} + w^*_{Nj}}{2} + \sum_{i=m+1}^{N-1} w^*_{ij}\right) \geq w^*_{Nj} - \gamma\int_{\xi}^{1} w(x,y)dx =$$

$$\frac{1}{4}\left(K(1-\gamma+\gamma\xi) + \left(-\frac{1}{4} + \frac{\gamma}{24} - \frac{\gamma}{3}\left(\xi - \frac{1}{2}\right)^3\right) - \left(y - \frac{1}{2}\right)^2(1-\gamma+\gamma\xi) + a\left(1 - \frac{\gamma}{2} + \frac{\gamma\xi^2}{2}\right)\right) \equiv \tilde{g}_{Nj}. \quad (57)$$

In formula (57) $\tilde{g}_{Nj} \leq g^*_{Nj}$. Thus, the condition $g^*_{Nj} > 1$ is stricter than the condition $\tilde{g}_{Nj} > 1$. However, we note that $\tilde{g}_{Nj} \to g^*_{Nj}$ when $h \to 0$. Changing the condition from g^*_{Nj} to $\tilde{g}_{Nj}$ makes sense, as by using the new condition in formula (57), the coefficients of a and K have a simple interpretation.

In Tables 1 and 2, we provide some obtained combinations of the parameters ξ_0, γ_0, and a, when the values are $g^*_{Nj} \geq 1$, guaranteeing that $w(x,y)$ is a majorant function for the error z_{ij}. Parameter γ_1 was calculated according to formula (56).

Table 1. Values of $\tilde{g}_{Nj} \geq 1$ at $\xi_0 = 0.8$ and $\gamma_1 = 6.7445$.

γ_0	$a = 3$	$a = 4$	$a = 5$
$\gamma_0 = 1.0$	0.811	1.016	1.221
$\gamma_0 = 2.0$	0.621	0.781	0.941
$\gamma_0 = 3.0$	0.432	0.547	0.662

Table 2. Values of $\tilde{g}_{Nj} \geq 1$ at $\xi_0 = 0.9$ and $\gamma_1 = 11.6599$.

γ_0	$a = 3$	$a = 4$	$a = 5$
$\gamma_0 = 1.0$	0.903	1.129	1.355
$\gamma_0 = 2.0$	0.805	1.008	1.210
$\gamma_0 = 3.0$	0.708	0.887	1.065

Numerical calculations show that $\tilde{g}_{Nj} \geq 1$ for higher values of parameter ξ_0 (the lower bound of the integral nonlocal conditions), i.e., ξ_0 approaches closer to 1. Additionally, it depends on γ_0 which is expected to be lower.

Increasing the value of γ_0 towards γ_1 always requires a large regulative parameter a, which leads us to a greater value of the error. In Table 3, the dependence of the condition $g^*_{Nj} \geq 1$ on the value of the parameter a can be seen.

From the results in Tables 1 and 2, we see that for relatively small γ values, the condition $\tilde{g}_{Nj} \geq 1$ can be ensured by increasing K and a. However, when γ values approach γ_1, (Tables 3 and 4), the situation changes for the worse. As the values of K and a increase, the condition $\tilde{g}_{Nj} > 1$ remains unfulfilled.

As in the expression of the $\tilde{g}_{Nj}$ coefficient to a

$$1 - \frac{\gamma_0}{2} + \frac{\gamma_0 \xi_0^2}{2} > 0, \text{ when } 0 < \gamma_0 < \frac{2}{1 - \xi_0^2},$$

then for the fixed values of ξ and γ, the value of $\tilde{g}_{Nj}$ increases as parameter a increases. Thus, for the fixed ξ_0 and γ_0, the function $w(x, y)$ will always be a majorant only if the coefficient to parameter a in (57) is positive. Parameter a increases without bound and $\gamma_0 < \overline{\gamma} = \dfrac{2}{1 - \xi_0^2}$.

Table 3. Values of $\tilde{g}_{Nj} \geq 1$ at $\xi_0 = 0.8$ and $\gamma_0 \to \gamma_1$.

γ_0 / a	$a = 4$	$a = 8$	$a = 42$	$a = 450$
$\gamma_0 = 1.0$	1.016	1.836	8.806	92.446
$\gamma_0 = 3.0$	0.547	1.007	4.917	51.837
$\gamma_0 = 5.0$	0.0078	0.178	1.028	11.228
$\gamma_0 = 5.5$	−0.045	−0.035	0.050	1.070
$\gamma_0 = 6.0$	−0.169	−0.249	−0.929	−9.089
$\gamma_0 = 6.5$	−0.292	−0.462	−1.907	−19.247

When $\gamma_0 \geq \overline{\gamma}$, the coefficient to parameter a becomes negative, and $w(x, y)$ is no longer a majorant, as can be seen in Tables 3 and 4.

Table 4. Values of $\tilde{g}_{Nj}$ ($\xi_0 = 0.9$), $\overline{\gamma} = 10.5263$, $\gamma_1 = 11.6599$.

γ_0 / a	$a = 3.5$	$a = 7$	$a = 28$	$a = 85$	$a = 1750$
$\gamma_0 = 1.0$	1.016	1.808	6.559	19.455	3.96161
$\gamma_0 = 5.0$	0.579	1.038	3.794	11.275	229.807
$\gamma_0 = 9.0$	0.141	0.268	1.030	3.096	63.452
$\gamma_0 = 10.0$	0.032	0.076	0.338	1.051	21.863
$\gamma_0 = 10,5$	−0.026	−0.024	−0.010	0.025	1.066
$\gamma_0 = 11.0$ $\gamma_0 \geq \overline{\gamma}$	−0.083	−0.123	−0.359	−1.000	−19.732

The conclusion follows from (57):

Conclusion 7: The larger the value γ_0, the larger the value of the parameter a, and thus the larger the value of C in estimating the error z_{ij} (see formula (55)).

8. Conclusions and Generalization

This paper analysed which values of the parameters ξ and γ make the matrix of the system of difference equations an M-matrix. By applying M-matrix theory, the convergence conditions of iterative methods for solving the system of nonlinear equations were determined. If the matrix of the difference equation system is an M-matrix, many iterative methods converge [18].

If the matrix of the system of difference equations is an M-matrix, then we can construct a majorant function to estimate the error in the solution for any parameter $\xi \in [0, 1)$. However, numerical results indicate that this is only true under a certain combination of the parameters ξ and γ. As the parameter γ increases, it is necessary to increase the parameter a as well, which leads to greater error. Numerical results show the same effect if we decrease the parameter ξ towards zero.

In expression (39) of the majorant function $w(x, y)$, we used the regulative member ax, which ensures that $w^*(x, y) \geq 0$.

The construction of the majorant (according to S. Gerschgorin [52–55]), based on M-matrix theory, is a suitable method for estimating the error in the finite difference method when solving boundary value problems for elliptic equations with nonlocal conditions.

We would like to provide a few more comments on the applications of M-matrix theory used in this paper for the investigation of difference schemes. The concept of the M-matrix [50] and the proof of convergence of the difference scheme by constructing the majorant [52] appeared around the same time in the fourth decade of the last century. In 1962, in the monograph of R. S. Varga [51], considering the application of Perron–Frobenius non-negative matrix theory to the investigation of difference schemes for elliptic and parabolic equations, M-matrices were already mentioned. However, the role of M-matrices in solving differential equations with classical Dirichlet and Neumann conditions was not substantial.

With the introduction of new mathematical models with nonlocal conditions, new research methods were needed. Here, it became clear that the theory of M-matrices is one of the suitable models for composing and analysing numerical methods for problems with nonlocal conditions.

Using the regular splitting principle [50,57], well known in linear algebra, it is possible to create convergent iterative methods for difference equation systems with nonlocal conditions [18]. However, the proof of the convergence of difference schemes based on the construction of a majorant has one shortcoming [54]—there is no algorithm for explicitly constructing the majorant. Nevertheless, when applying the theory of M-matrices in this paper, the situation is better. According to the properties of M-matrices, a vector $x > 0$ exists such that $Ax > 0$ for an M-matrix A. It is not difficult to verify that, from there, it follows that the majorant exists.

The method of proof for the convergence of the difference schemes provided in this paper is also applicable to many other equations, boundary conditions, or numerical methods—for instance, solving the system of Equations (1)–(3) using the finite element method with linear elements. It is important that it is possible to formulate the difference problem with an M-matrix (one such problem is the Poisson equation with an integral Neumann condition). Of course, the M-matrix method is not universal (for example, it is not suitable for problem (1)–(3) when $\gamma < 0$), but its potential is far from exhausted.

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