KAUNAS UNIVERSITY OF TECHNOLOGY

INGA TELKSNIENĖ

ALGEBRAIC TRANSFORMATIONS FOR THE ANALYSIS OF FRACTIONAL DIFFERENTIAL EQUATIONS

Doctoral dissertation
Natural Sciences, Informatics (N 009)

2024, Kaunas

This doctoral dissertation was prepared at Kaunas University of Technology, Faculty of Mathematics and Natural Sciences, Department of Mathematical Modelling during the period of 2020–2024. The studies were supported by the Research Council of Lithuania.
The doctoral right has been granted to Kaunas University of Technology together with Vytautas Magnus University and Vilnius Gediminas Technical University (VILNIUS TECH).

Scientific Supervisor:
Prof. Dr. Hab. Minvydas Kazys RAGULSKIS (Kaunas University of Technology, Natural sciences, Informatics, N 009).

The dissertation was edited by: English language editor Dr. Armandas Rumšas (Publishing House *Technologija*), Lithuanian language editor Aurelija Gražina Rukšaitė (Publishing House *Technologija*).

The official defence of the dissertation will be held at 10 a.m. on 11 June, 2024 at the public meeting of the Dissertation Defence Board of Informatics Science Field in Rectorate Hall at Kaunas University of Technology.

Address: K. Donelaičio 73-402, LT-44249 Kaunas, Lithuania.
Phone (+370) 608 28 527; email doktorantura@ktu.lt

The doctoral dissertation was sent out on 10 May, 2024.
The doctoral dissertation is available on the internet at http://ktu.edu and at the libraries of Kaunas University of Technology (Gedimino 50, LT-44239 Kaunas, Lithuania), Vytautas Magnus University (K. Donelaičio 52, LT-44244 Kaunas, Lithuania) and Vilnius Gediminas Technical University (VILNIUS TECH) (Saulėtekio 14, LT-10223, Vilnius, Lithuania)

KAUNO TECHNOLOGIJOS UNIVERSITETAS

INGA TELKSNIENĖ

ALGEBRINĖS TRANSFORMACIJOS TRUPMENINĖS EILĖS DIFERENCIALINIŲ LYGČIŲ TYRIMUI

Daktaro disertacija
Gamtos mokslai, informatika (N 009)

2024, Kaunas

Disertacija rengta 2020–2024 metais Kauno technologijos universiteto Matematikos ir gamtos mokslų fakultete, Matematinio modeliavimo katedroje. Mokslinius tyrimus rėmė Lietuvos mokslo taryba.
Doktorantūros teisė Kauno technologijos universitetui suteikta kartu su Vytauto Didžiojo universitetu ir Vilniaus Gedimino technikos universitetu (VILNIUS TECH).

Mokslinis vadovas:
prof. habil. dr. Minvydas Kazys RAGULSKIS (Kauno technologijos universitetas, gamtos mokslai, informatika, N 009).

Disertaciją redagavo: anglų kalbos redaktorius dr. Armandas Rumšas (leidykla „Technologija“), lietuvių kalbos redaktorė Aurelija Gražina Rukšaitė (leidykla „Technologija“).

Informatikos mokslo krypties disertacijos gynimo taryba:
prof. habil. dr. Rimantas BARAUSKAS (Kauno technologijos universitetas, gamtos mokslai, informatika, N 009) – **pirmininkas**;
prof. habil. dr. Antanas ČENYS (Vilniaus Gedimino technikos universitetas (VILNIUS TECH), gamtos mokslai, informatika, N 009);
prof. dr. Gintaras PALUBECKIS (Kauno technologijos universitetas, gamtos mokslai, informatika, N 009);
prof. dr. Miguel A. F. SANJUAN (Rey Juan Carlos universitetas, Ispanija, gamtos mokslai, informatika, N 009);
doc. dr. Vadimas STARIKOVIČIUS (Vilniaus Gedimino technikos universitetas (VILNIUS TECH), gamtos mokslai, informatika, N 009).

Disertacija bus ginama viešame Informatikos mokslo krypties disertacijos gynimo tarybos posėdyje 2024 m. birželio 11 d. 10 val. Kauno technologijos universiteto Rektorato salėje.

Adresas: K. Donelaičio g. 73-402, LT-44249 Kaunas, Lietuva.
Tel. (+370) 608 28 527; el. paštas doktorantura@ktu.lt

Disertacija išsiųsta 2024 m. gegužės 10 d.
Su disertacija galima susipažinti interneto svetainėje http://ktu.edu, Kauno technologijos universiteto bibliotekoje (Gedimino g. 50, LT-44239 Kaunas, Lietuva), Vytauto Didžiojo universiteto bibliotekoje (K. Donelaičio g. 52, LT-44244, Kaunas, Lietuva) ir Vilniaus Gedimino technikos universiteto (VILNIUS TECH) bibliotekoje (Saulėtekio al. 14, LT-10223, Vilnius, Lietuva)

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF ABBREVIATIONS

FDE – fractional differential equation;
CFDE – Caputo fractional differential equation;
CTDL – Caputo trupmeninės eilės diferencialinė lygtis.

INTRODUCTION

Caputo Fractional Differential Equations (CFDEs) have recently emerged as an important tool for modeling complex phenomena in a variety of scientific fields owing to their ability to model systems exhibiting memory or hereditary properties. The extensive applicability of Caputo's differential equations necessitates their exploration via both analytical and numerical techniques, thus making it a highly relevant task. Thus, **the main objective** of this thesis is to develop a novel semi-analytical framework for the construction and analysis of solutions to Caputo fractional differential equations by utilizing the concepts of Caputo algebra of fractional power series, as presented in [1; 2].

The realization of this objective was achieved by completing the following **tasks**:

1. Development of a methodology for the construction of fractional power series solutions to a $\left({}^{C}\mathrm{D}^{\left(\frac{1}{n}\right)}\right)^{n}$ type Riccati CFDE and investigation of the structure of such solutions.
2. Development of the methodology for the construction of fractional power series solutions to a more general class of equations – $\left({}^{C}\mathrm{D}^{\left(\frac{1}{n}\right)}\right)^{n}$ type CFDEs with polynomial nonlinearity.
3. Development of the analytical framework for the extension of solutions to CFDEs with polynomial nonlinearity to the negative half-line and investigation of the properties of such an extension.
4. Creation of a semi-analytical scheme for the construction of approximate solutions to CFDEs.
5. Development of a methodology for the construction of fractional power series solutions to an even wider class of nonlinear CFDEs – ${}^{C}\mathrm{D}_{x}^{\left(\frac{1}{n}\right)}$ type CFDEs.

This doctoral dissertation is based on a collection of scientific papers, each of which fulfilled one or several tasks outlined above. The first paper *The Fractal Structure of Analytical Solutions to Fractional Riccati Equation* paved the way for this research by developing a novel methodology for the construction of fractional power series solutions to a specific Riccati-type CFDE. The presented results were expanded upon significantly in the paper *The Extension of Analytic Solutions to FDEs to the Negative Half-Line* by demonstrating that a refined methodology can be utilized not only for Riccati-type CFDEs, but also for a wider range of equations including CFDEs with polynomial nonlinearity. The next step, published in paper *An Operator-Based Scheme for the Numerical Integration of FDEs*, was concerned with using the developed methodology to create a semi-analytical scheme aimed towards the construction of approximate solutions to CFDEs. The final paper presented in this thesis, titled *The Construction of Solutions to ${}^{C}\boldsymbol{D}^{(1/n)}$ Type FDEs via Reduction to $\left({}^{C}\boldsymbol{D}^{\left(\frac{1}{n}\right)}\right)^{n}$ Type FDEs*, brought all the previous research together to develop a comprehensive approach for the construction of solutions to an even wider class of nonlinear CFDEs.

This thesis contributes to the fields of Mathematics and Informatics by bridging theoretical mathematics with practical computational applications. At the core of this research is the application of symbolic computations to handle the extensive mathematical expressions that arise during the analysis of CFDEs. Thus, the methodologies developed herein are not just theoretical derivations, but they also serve as practical executable algorithms enabling the effective use of computer algebra systems in solving and analyzing Caputo fractional differential equations.

In all the publications that make up this thesis, the *Matlab* numeric computing platform was employed for conducting numerical experiments, while *Maple* and *Mathematica* computer algebra systems were utilized for symbolic computations.

Co-authors' contribution to papers

A list of contributions from the co-authors of the papers included in this thesis is presented below.

1. *The Fractal Structure of Analytical Solutions to Fractional Riccati Equation* by Zenonas Navickas, Tadas Telksnys, Inga Timofejeva (now Telksnienė), Romas Marcinkevicius, and Minvydas Ragulskis
    a. Z. Navickas devised the idea for the construction of solutions to specific $\left({}^{C}\mathbf{D}^{\left(\frac{1}{n}\right)}\right)^{n}$ type FDEs via their reduction to the characteristic ODEs, and formulated the preliminary versions of the necessary mathematical derivations.
    b. T. Telksnys expanded upon the concepts formulated by Z. Navickas, contributed to the formal analysis of the structure of solutions to the analyzed equation, and corresponded with the editorial office of the journal.
    c. I. Timofejeva (now Telksnienė) formalized and generalized the ideas about the nested structure of solutions to CFDEs, executed most of the necessary computer algebra and numerical computations, and wrote the manuscript text.
    d. R. Marcinkevičius assisted I. Telksnienė with symbolic computations, and contributed to the reviewing and editing of the manuscript.
    e. M. Ragulskis supervised the entire research process, organized weekly seminars, provided valuable advice on the structure of the study, and oversaw the writing of the manuscript.
2. *The Extension of Analytic Solutions to FDEs to the Negative Half-Line* by Inga Timofejeva (now Telksnienė), Zenonas Navickas, Tadas Telksnys, Romas Marcinkevičius, Xiao-Jun Yang, and Minvydas Ragulskis
    a. I. Timofejeva (now Telksnienė) formalized and generalized the ideas related to the extension of solutions to CFDEs to the negative half-

line, carried out most of the necessary computer algebra and numerical computations, and prepared the manuscript text.
   b. Z. Navickas conceived the idea to expand the existing methodology to a more general class of equations, and drafted the preliminary version of the analytical framework for the extension of solutions to such CFDEs to the negative half-line.
   c. T. Telksnys expanded upon the concepts formulated by Z. Navickas, contributed to the development of the presented methodology, and corresponded with the editorial office of the journal.
   d. R. Marcinkevičius assisted I. Telksnienė with symbolic computations, and contributed to the reviewing and editing of the manuscript.
   e. X.-J. Yang contributed to the development of the presented methodology, and participated in the review and refinement of the manuscript.
   f. M. Ragulskis oversaw the entire research process, organized weekly seminars, provided valuable advice on the structure of the study, and supervised the writing of the manuscript.
3. *An Operator-Based Scheme for the Numerical Integration of FDEs* by Inga Timofejeva (now Telksnienė), Zenonas Navickas, Tadas Telksnys, Romas Marcinkevičius, and Minvydas Ragulskis
   a. I. Timofejeva (now Telksnienė) formalized the preliminary version of the operator-based scheme for the numerical integration of FDEs, performed most of the symbolic and numerical calculations, and was responsible for writing the manuscript text.
   b. Z. Navickas contributed to the formal analysis of the presented concepts.
   c. T. Telksnys contributed to the development and refinement of the presented techniques.
   d. R. Marcinkevičius performed the preliminary computations necessary to showcase the paper's concept, and contributed to the reviewing and editing of the manuscript.
   e. M. Ragulskis suggested the topic of this study, oversaw the whole research process, organized weekly seminars, provided valuable advice on the structure of the paper, and supervised the writing of the manuscript.
4. *The Construction of Solutions to ${}^{C}\boldsymbol{D}^{(1/n)}$ type FDEs via Reduction to $\left({}^{C}\boldsymbol{D}^{\left(\frac{1}{n}\right)}\right)^{n}$ Type FDEs* by Romas Marcinkevičius, Inga Telksnienė, Tadas Telksnys, Zenonas Navickas, and Minvydas Ragulskis
   a. R. Marcinkevičius performed the preliminary symbolic computations necessary to illustrate the paper's ideas, and contributed to the reviewing and editing of the manuscript.

b. I. Telksnienė formalized and generalized the ideas of Z. Navickas and T. Telksnys, assisted R. Marcinkevičius with computer algebra computations, prepared all the necessary computational examples, drafted and refined the manuscript text, and corresponded with the editorial office.
c. T. Telksnys expanded upon the concepts formulated by Z. Navickas, and contributed to the development of the presented methodology.
d. Z. Navickas conceived the idea to expand the methodologies for the construction of fractional power series solutions presented in the previous papers to a more general class of equations, and drafted the preliminary mathematical derivations.
e. M. Ragulskis oversaw the entire research process, organized weekly seminars, provided valuable insights about the structure of the study, and supervised the writing and editing of the manuscript.

It should be noted that all the co-authors have been informed of and have authorized the inclusion of these publications in this thesis. It is also of note that all the papers listed above were published as Open Access; thus, it is unnecessary to obtain the permission from the publishers to reprint these papers.

1. LITERATURE REVIEW

Differential equations have long been a cornerstone in the field of mathematics, by providing a powerful tool for modeling and analyzing various phenomena in numerous scientific disciplines including physics, engineering, biology, economics, etc.

In recent years, a new class of differential equations, known as *Fractional Differential Equations* (FDEs), has garnered significant attention in the scientific community. Unlike the 'classical' differential equations which involve integer-order derivatives, FDEs employ fractional-order derivatives, thus extending the concept of differentiation to non-integer orders. This extension has proven to be particularly useful in describing complex systems with memory or hereditary properties. The relevance of fractional differential equations in modern science is underscored by their application in various areas ranging from the classical FDEs in viscoelasticity [3], to more novel physical and engineering fields [4-6], and beyond to biology [7], medicine [8], and economics [9].

A review of the main concepts and applications of fractional calculus is presented in the subsequent sections.

1.1. Introduction to Fractional Derivatives

The concept of a fractional derivative and, in turn, fractional calculus originated in the 17th century, with initial discussions documented in the correspondence between Leibniz and L'Hospital in 1695 [10]. The main idea was to extend the definition of a classical integer-order derivative to non-integer orders, by satisfying the criteria outlined below.

Let $\mathbf{D}_x^{(n)} = \frac{\mathrm{d}^n}{\mathrm{d}x^n}$ denote a classical differential operator of integer order n and $f(x)$ be an arbitrary function. Then, the fractional derivative is an operator $\widehat{\mathbf{D}}_x$, such that:

1. $\widehat{\mathbf{D}}_x^{(n)} f(x) = \mathbf{D}_x^{(n)} f(x) = \frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}$, for $n \in \mathbb{N}_0$, i.e., fractional and classical integer-order derivatives coincide if the order of differentiation is a nonnegative integer.
2. $\widehat{\mathbf{D}}_x^{(\alpha)} f(x)$ can be computed if α is not an integer.

Naturally, constructing a fractional derivative according to such a broad definition can yield multiple non-equivalent operators. Indeed, since the advent of fractional derivatives, over twenty distinct types have been formulated, each characterized by unique properties and specific fields of application.

1.2. Definitions of Fractional Derivatives

Three most commonly used definitions of fractional derivatives and their properties are outlined in this section.

Let $f(\cdot)$ denote an arbitrary function and $\Gamma(\cdot)$ denote a Gamma function.

Riemann-Liouville (≈1850) fractional differentiation operator of order $\alpha \in \mathbb{R}$ ($\alpha \geq 0$) is denoted as ${}^{RL}\mathbf{D}_x^{(\alpha)}$ and is defined as follows [11]:

$$ {}^{RL}\mathbf{D}_x^{(\alpha)} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_0^x \frac{f(\tau)}{(x-\tau)^{\alpha-n+1}} \mathrm{d}\tau, \tag{1.2.1} $$

where $n = \lceil \alpha \rceil$.

Caputo (1967) fractional differentiation operator of order $\alpha \in \mathbb{R}$ ($\alpha \geq 0$) is denoted as ${}^{C}\mathbf{D}_x^{(\alpha)}$ and is defined as follows [12]:

$$ {}^{C}\mathbf{D}_x^{(\alpha)} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(\tau)}{(x-\tau)^{\alpha-n+1}} \mathrm{d}\tau, \tag{1.2.2} $$

where $n = \lceil \alpha \rceil$.

We note that integration and n-th order differentiation operations are reversed in Caputo's definition compared to Riemann-Liouville's.

Grünwald-Letnikov (1868) fractional differentiation operator of order $\alpha \in \mathbb{R}$ ($\alpha \geq 0$) is denoted as ${}^{GL}\mathbf{D}_x^{(\alpha)}$ and is defined as follows [11]:

$$ {}^{GL}\mathbf{D}_x^{(\alpha)} f(x) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{j=0}^{x/h} (-1)^j \binom{\alpha}{j} f(x - jh) \tag{1.2.3} $$

where $\binom{\alpha}{j} = \frac{\Gamma(\alpha+1)}{j!\,\Gamma(\alpha-j+1)}$ denotes binomial coefficient.

We note that Grünwald-Letnikov definition is a discretization of the Riemann-Liouville fractional derivative. Thus, it provides a straightforward way to approximate fractional derivatives for numerical computations.

An important property that all three fractional differentiation operators presented above have in common is ***non-locality***, which means that the value of the fractional derivative at a certain point is influenced by the function's values at all the points in the past. This is in contrast to the classical integer-order derivatives, which are local operators, which means that the value of the derivative at a point depends only on the function's values in a very small neighborhood around that point. This non-locality property facilitates the modeling of the so-called memory effects in various systems, where the current state is significantly influenced by a historical series of states, rather than just by the immediate past.

The Caputo fractional derivative is currently one of the most popular choices in terms of modelling due to several convenient properties that it possesses. *Fractional*

Differential Equations (FDEs) involving many other fractional derivatives necessitate the specification of fractional order initial conditions, which can complicate the modeling process. In contrast, Caputo FDEs (CFDEs) only require classical integer-order initial conditions, thereby simplifying the application considerably [13]. Additionally, contrary to some other fractional derivatives (for example, Riemann-Liouville's), the Caputo derivative of a constant is zero, which aligns well with certain practical applications, thus offering a more intuitive modeling approach. Due to the notable usage of the Caputo fractional derivative in recent research, this study shall mainly focus on examining the Caputo fractional derivative, and, subsequently, Caputo fractional differential equations.

1.3. Caputo Fractional Differential Equations

As mentioned above, *Caputo Fractional Differential Equations* (CFDEs) have recently emerged as a powerful tool in the mathematical modeling of various complex systems, exhibiting memory or hereditary properties. Successful real-world applications of CFDEs can be seen in the fields of physics (e.g., plasma physics [14], optics [15]), engineering (e.g., viscoelasticity [16]), biology (e.g., natural systems [17], environmental engineering [18]), economics and finance (e.g., competition systems [19]), health sciences (e.g., epidemiology [20], neuroscience [21]), etc.

When it comes to the analysis and solution of CFDEs, several mathematical methods have been developed which could be separated into three general groups:

1. **Analytical methods.** These methods provide exact analytical solutions. Most common examples include integral transform methods, such as Laplace or Fourier transform methods [22; 23]. Notably, I. Podlubny pioneered the application of the Laplace transform with Mittag-Leffler functions to address a wide array of initial value problems for fractional differential equations [24]. Analytical methods are typically tailored to address specific types of problems and conditions, while lacking a generalized approach. This means that each new kind of CFDE might necessitate the development of a unique analytical strategy. Also, naturally, the complexity escalates considerably when these methods are applied to non-linear or higher-order problems, often to the point when finding exact solutions is intractable.
2. **Semi-analytical methods.** Such methods are a hybrid approach which employs both analytical and numerical strategies. They seek to find approximate solutions but in the form of functional or series representations, which can be further analyzed and manipulated analytically. Most commonly used methods in this category include Variational Iteration [25] and Adomian Decomposition [26] methods and their modifications.
3. **Numerical methods.** These methods involve approximating solutions using numerical algorithms and provide solutions at discrete points in the domain. Popular methods under this category include the Finite Difference Method

[27], the Finite Element Method [28], Spectral Methods [29] and their modifications.

The aim of this thesis is to develop a semi-analytical methodology for the analysis of CFDEs through algebraic transformations, by utilizing the concepts of Caputo algebra of fractional power series, as presented in [1; 2]. The main definitions related to this approach are outlined in the following section.

1.4. Caputo Algebra of Fractional Power Series

This section outlines the main concepts of Caputo fractional power series which shall be utilized in further sections and which are necessary for the understanding of the thesis papers.

Let the order of the Caputo fractional derivative be denoted as $\alpha = \frac{k}{n}$, where $k, n \in \mathbb{N}$ and $\gcd(k, n) = 1$. Also, let $x \geq 0$.

For the remainder of this thesis, we shall consider functions expressed via Caputo fractional power series, i.e., power series that are summed over fractional powers:

$$f(x) = \sum_{j=0}^{+\infty} v_j w_j^{(n)}, \tag{1.4.1}$$

where $v_j \in \mathbb{C}$ are coefficients of the series and $w_j^{(n)}, n \in \mathbb{N}, j = 0,1, \ldots$ are the fractional power series basis functions of order n defined as:

$$w_j^{(n)} = \frac{x^{\frac{j}{n}}}{\Gamma\left(\frac{j}{n} + 1\right)}, j = 0,1, \ldots \tag{1.4.2}$$

The set of Caputo fractional power series with respect to parameter n is denoted as follows:

$$^C\mathbb{F}_n = \left\{ \sum_{j=0}^{+\infty} v_j w_j^{(n)}; \quad v_j \in \mathbb{C} \right\}. \tag{1.4.3}$$

Let $f_1(x) = \sum_{j=0}^{+\infty} a_j w_j^{(n)}, f_2(x) = \sum_{j=0}^{+\infty} b_j w_j^{(n)} \in {}^C\mathbb{F}_n$. Standard operations of addition, product by a scalar and multiplication of functions in ${}^C\mathbb{F}_n$ are defined as:

$$f_1(x) + f_2(x) = \sum_{j=0}^{+\infty} \left(a_j + b_j\right) w_j^{(n)}; \tag{1.4.4}$$

$$A \cdot f_1(x) = \sum_{j=0}^{+\infty} A a_j w_j^{(n)}, A \in \mathbb{C}; \tag{1.4.5}$$

$$f_1(x) \cdot f_2(x) = \sum_{j=0}^{+\infty} \left(\sum_{k=0}^{j} \binom{\frac{j}{n}}{\frac{k}{n}} a_k b_{j-k} \right) w_j^{(n)}, \tag{1.4.6}$$

where $\binom{\frac{j}{n}}{\frac{k}{n}} = \frac{\Gamma\left(\frac{j}{n}+1\right)}{\Gamma\left(\frac{k}{n}+1\right)\Gamma\left(\frac{j}{n}-\frac{k}{n}+1\right)}$ denotes a binomial coefficient.

Thus, the set ${}^{C}\mathbb{F}_n$ with the standard addition, product by a scalar and multiplication operations forms an algebra over $\mathbb{C}$. This algebra is called the Caputo algebra and is denoted as follows:

$$ {}^{C}\mathcal{F}_n = \langle {}^{C}\mathbb{F}_n; +, \cdot \,|\mathbb{C}\rangle \tag{1.4.7}$$

The Caputo fractional differentiation operator of order $\frac{1}{n}$ of the basis functions $w_j^{(n)}$ is defined as:

$$ {}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)} w_j^{(n)} = \begin{cases} 0, & j = 0 \\ w_{j-1}^{(n)}, & j = 1,2,\ldots \end{cases} \tag{1.4.8}$$

Therefore, the Caputo fractional derivative of order $\alpha = \frac{k}{n}$ of a function $f(x) = \sum_{j=0}^{+\infty} v_j w_j^{(n)} \in {}^{C}\mathbb{F}_n$ reads:

$$ {}^{C}\mathbf{D}_x^{\left(\frac{k}{n}\right)} f(x) = \left({}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)}\right)^k f(x) = \sum_{j=0}^{+\infty} v_{j+k} w_j^{(n)} \in {}^{C}\mathbb{F}_n. \tag{1.4.9}$$

We note that this definition of the Caputo differentiation operator is congruent with the original integral-based definition (1.2.2) [2].

This approach has already been applied for the construction of fractional power series solutions to linear fractional differential equations [2], as well as equations of the following type [1]:

$$ x\left({}^{C}\mathbf{D}_x^{\left(\frac{1}{2}\right)}\right)^2 y(x) = B_0 + B_1 y(x) + B_2 y(x)^2; \;\; B_0, B_1, B_2 \in \mathbb{R}. \tag{1.4.10}$$

The main objective of this thesis is to expand the Caputo algebra-based framework for the analysis and construction of solutions to a wider range of fractional differential equations.

2. REVIEW OF PAPERS

2.1. Review of *The Fractal Structure of Analytical Solutions to Fractional Riccati Equation*

2.1.1. Paper details

Title: The Fractal Structure of Analytical Solutions to Fractional Riccati Equation

Authors: Zenonas Navickas, Tadas Telksnys, **Inga Timofejeva (now Inga Telksnienė)**, Romas Marcinkevičius, and Minvydas Ragulskis

To be cited as: Navickas, Zenonas, et al. The Fractal Structure of Analytical Solutions to Fractional Riccati Equation. *Fractals*, doi: 10.1142/S0218348X23401308

Input from I.Telksnienė: I. Telksnienė formalized and generalized the ideas about the nested structure of solutions to CFDEs, executed most of the necessary computer algebra and numerical computations, and wrote the manuscript text.

2.1.2. Summary of the paper

Objective of the paper

This paper aims to:

1. Introduce a novel methodology for the construction of solutions to the fractional Riccati equation of the following form:
$$\left({}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)}\right)^n y(x) = a_2 y(x)^2 + a_1 y(x) + a_0;\ \ a_0, a_1, a_2 \in \mathbb{C};\ n \in \mathbb{N},$$
2. Investigate the structure of the fractional power series solutions to the fractional Riccati equation.

Methodology and results

The following initial value problem for the fractional Riccati equation is considered:

$$\left({}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)}\right)^n y_n = a_2 y_n^2 + a_1 y_n + a_0; \tag{2.1.1}$$

$$\left.\left({}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)}\right)^k y_n\right|_{x=0} = s_k^{(n)}; k = 0, \dots, n-1, \tag{2.1.2}$$

where $a_0, a_1, a_2 \in \mathbb{C}$, $n \in \mathbb{N}$, $y_n = y_n\left(x; s_0^{(n)}, s_1^{(n)}, \dots, s_{n-1}^{(n)}\right) \in {}^{C}\mathbb{F}_n$, and the parameters $s_0^{(n)}, s_1^{(n)}, \dots, s_{n-1}^{(n)}$ correspond to the initial conditions formulated at $x = 0$.

It is important to note that the operator $\left({}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)}\right)^n$ is not identical to $\frac{\mathrm{d}}{\mathrm{d}x}$ for $n > 1$, since the former operator acts on Caputo fractional power series (1.4.1) consisting of non-integer powers of x, while the operator $\frac{\mathrm{d}}{\mathrm{d}x}$ is applied to classical Taylor power series, comprised of only integer powers of x.

Let:

$$y_n = \sum_{j=0}^{+\infty} c_j w_j^{(n)} = \sum_{j=0}^{+\infty} \gamma_j^{(n)} \Gamma\left(\frac{j}{n} + 1\right) w_j^{(n)} = \sum_{j=0}^{+\infty} \gamma_j^{(n)} x^{\frac{j}{n}};\ c_j \in \mathbb{C}. \qquad (2.1.3)$$

Firstly, recurrence relations for the coefficients $\gamma_j^{(n)}$ of the solution to the fractional Riccati equation are derived by inserting (2.1.3) into (2.1.1)-(2.1.2) and performing various algebraic manipulations on the obtained expressions. The resulting relations are as follows:

$$\gamma_k^{(n)} = \frac{s_k^{(n)}}{\Gamma\left(\frac{k}{n} + 1\right)};\ k = 0,1,\dots,n-1; \qquad (2.1.4)$$

$$(j+n)\gamma_{j+n}^{(n)} = n\left(a_2 \sum_{r=0}^{j} \left(\gamma_r^{(n)} \gamma_{j-r}^{(n)}\right) + a_1 \gamma_j^{(n)} + \delta_j a_0\right); j = 0,1,\dots, \qquad (2.1.5)$$

where $\delta_j = 1$ if $j = 0$, and $\delta_j = 0$ otherwise.

Next, the characteristic function of the sequence $\left(\gamma_j^{(n)}; j = 0,1,\dots\right)$ is defined as:

$$\varphi_n(t) = \sum_{j=0}^{+\infty} \gamma_j^{(n)} t^j. \qquad (2.1.6)$$

The following ordinary differential equation with respect to the generating function $\varphi_n(t)$ is then derived from the recurrence relation (2.1.5):

$$\frac{\mathrm{d}\varphi_n}{\mathrm{d}t} = nt^{n-1}(a_2\varphi_n^2(t) + a_1\varphi_n(t) + a_0) + \sum_{j=1}^{n-1} j\gamma_j^{(n)} t^{j-1}. \qquad (2.1.7)$$

It is demonstrated that the function $\varphi_n(t)$ can be utilized to express solutions to (2.1.1)-(2.1.2), since:

$$y_n = \sum_{j=0}^{+\infty} \gamma_j^{(n)} x^{\frac{j}{n}} = \varphi_n(\sqrt[n]{x}). \qquad (2.1.8)$$

Thus, the initial value problem on the Riccati fractional differential equation (2.1.1)-(2.1.2) is equivalent to the initial value problem on the following ordinary differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}(y_n - \psi_n) = a_2 y_n^2 + a_1 y_n + a_0; \qquad (2.1.9)$$

$$y_n(0) = s_0^{(n)} = \gamma_0^{(n)}, \qquad (2.1.10)$$

where $\psi_n(x) = \sum_{j=0}^{n-1} \gamma_j^{(n)} x^{\frac{j}{n}}$.

Naturally, the obtained ODE can be analyzed and integrated with any classical analytical or numerical techniques, thus providing a way to study this type of FDEs without the need for new methods.

Investigation of the structure of (2.1.9) and (2.1.5) leads to the observation that if the initial conditions of (2.1.1)-(2.1.2) are set to $s_1^{(n)} = s_2^{(n)} = \cdots = s_{n-1}^{(n)} = 0$, then $\psi_n(x) = 0$ and $\gamma_j^{(n)} \neq 0$ only if $j = kn, k \in \mathbb{N}$, which means that the solution y_n belongs to the set ${}^{C}\mathbb{F}_1$ and satisfies the classical ordinary Riccati equation:

$$\frac{\mathrm{d}y_n}{\mathrm{d}x} = a_2 y_n^2 + a_1 y_n + a_0. \tag{2.1.11}$$

Thus, for any $n \in \mathbb{N}$, the fractional Riccati equation (2.1.1) admits all solutions to the ODE (2.1.11) (if the relation $s_1^{(n)} = s_2^{(n)} = \cdots = s_{n-1}^{(n)} = 0$ holds), but has more solutions unique to it (if the relation $s_1^{(n)} = s_2^{(n)} = \cdots = s_{n-1}^{(n)} = 0$ does not hold).

Computational experiments verifying the analytical results presented above are provided; they use the following fractional Riccati equation as an example:

$$\left({}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)}\right)^n y_n = y_n^2 + y_n - 6. \tag{2.1.12}$$

The following difference measure is introduced in order to compare numerical solutions to (2.1.12) for different values of n:

$$\Delta_{n,m}\left(s_1^{(n)}, \dots, s_{n-1}^{(n)}; s_1^{(m)}, \dots, s_{m-1}^{(m)}\right) = \sum_{j=0}^{N} \left(\hat{y}_n\left(jh; s_1^{(n)}, \dots, s_{n-1}^{(n)}\right) - \hat{y}_m\left(jh; s_1^{(m)}, \dots, s_{m-1}^{(m)}\right)\right)^2, \tag{2.1.13}$$

where $\hat{y}_k\left(x; s_1^{(k)}, \dots, s_{k-1}^{(k)}\right)$ is the numerical solution to the fractional Riccati equation (2.1.12) of order k with the initial conditions $s_1^{(k)}, \dots, s_{k-1}^{(k)}$; h and N are the step-size and the number of steps of the Runge-Kutta 4th order numerical integrator, respectively. The initial conditions $s_0^{(n)}$ and $s_0^{(m)}$ are set to be equal. Fig. 2.1.1 depicts the plot of $\Delta_{1,2}\left(s_1^{(2)}\right)$ when $s_0^{(1)} = s_0^{(2)} = 0.5$. It can be seen that the solution to the fractional Riccati equation (2.1.12) of order 2 coincides with the solution of the non-fractional Riccati equation ($n = 1$) when $s_1^{(2)} = 0$. Fig. 2.1.2 shows the plots of $\Delta_{1,3}\left(s_1^{(3)}, s_2^{(3)}\right)$ and $\Delta_{1,3}\left(0, s_2^{(3)}\right)$ when $s_0^{(1)} = s_0^{(3)} = 0.5$. Analogously, it can be observed that the solution to the fractional Riccati equation (2.1.12) of order 3 coincides with the solution of the non-fractional Riccati equation ($n = 1$) when $s_1^{(3)} = s_2^{(3)} = 0$. A plot of solutions y_1 ($n = 1$) and y_3 ($n = 3$) not on the minimum point (0,0) is displayed in Fig. 2.1.3.

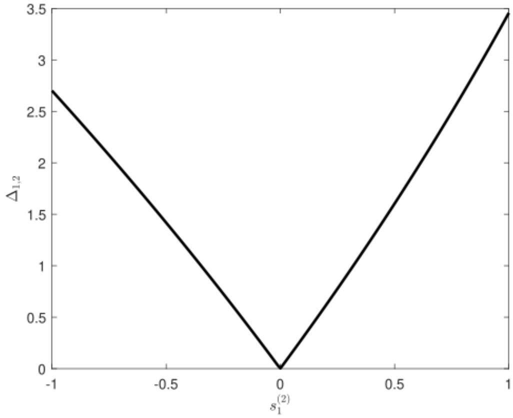


Fig. 2.1.1. Plot of $\boldsymbol{\Delta_{1,2}}\left(\boldsymbol{s_1^{(2)}}\right)$ for $\boldsymbol{s_0^{(1)}} = \boldsymbol{s_0^{(2)}} = \boldsymbol{0.5}$

(a)

(b)

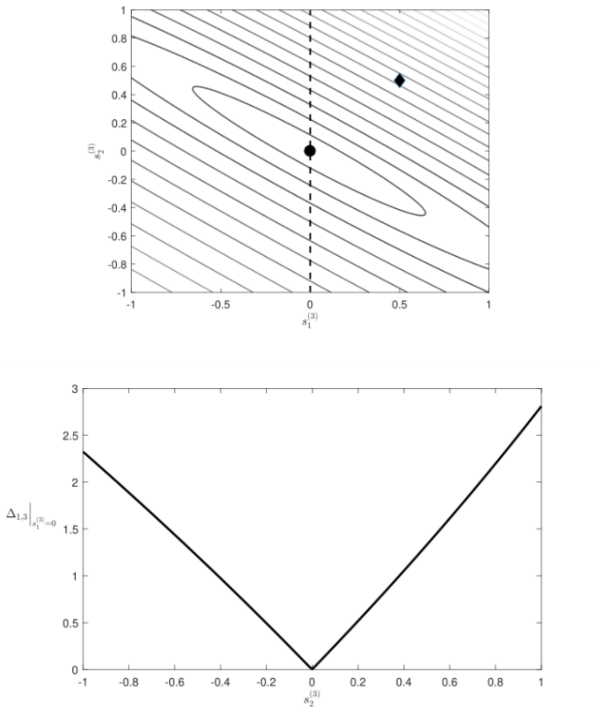


Fig. 2.1.2. Part (a) depicts the contour plot of $\boldsymbol{\Delta_{1,3}}\left(\boldsymbol{s_1^{(3)}}, \boldsymbol{s_2^{(3)}}\right)$ for $\boldsymbol{s_0^{(1)}} = \boldsymbol{s_0^{(2)}} = \boldsymbol{0.5}$. The black circle denotes the minimum point $\boldsymbol{s_1^{(3)}} = \boldsymbol{s_2^{(3)}} = \boldsymbol{0}$, where $\boldsymbol{\Delta_{1,3}} = \boldsymbol{0}$. The dashed line corresponds to the plot of $\boldsymbol{\Delta_{1,3}}\left(\boldsymbol{0}, \boldsymbol{s_2^{(3)}}\right)$ depicted in part (b). The diamond corresponds to to the initial conditions used in Fig. 2.1.3

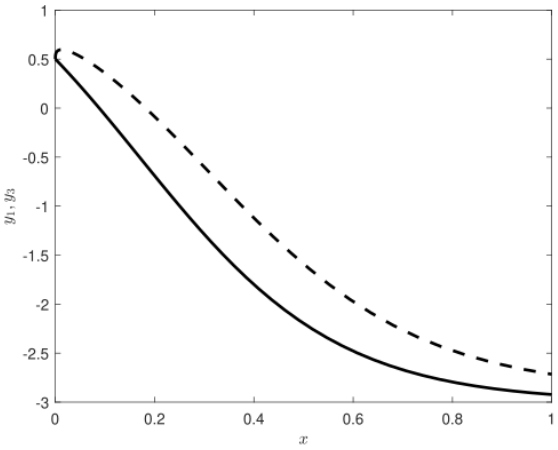


Fig. 2.1.3. Plot of solutions $\boldsymbol{y_1}$ (solid line) and $\boldsymbol{y_3}$ (dashed line) for $\boldsymbol{s_0^{(1)} = s_0^{(2)} = s_1^{(3)} = s_2^{(3)} = 0.5}$

The aforementioned observations about the nested structure of the solutions to the fractional Riccati equation can be extended further: any fractional Riccati equation (2.1.1) of order $n = km$; $k, m \in \mathbb{N}$ inherits solutions from the fractional Riccati equation of orders $n = k$ and $n = m$.

Let a, b be coprime natural numbers and $m \in \mathbb{N}$. The following properties can be derived from the definition of the Caputo fractional power series (see Section 1.4):

$$ {}^{C}\mathbb{F}_{am} \cap {}^{C}\mathbb{F}_{bm} = {}^{C}\mathbb{F}_{m}; \tag{2.1.14}$$

$$ {}^{C}\mathbb{F}_{am} \cup {}^{C}\mathbb{F}_{bm} \subseteq {}^{C}\mathbb{F}_{abm}. \tag{2.1.15}$$

Fig. 2.1.4 illustrates the relationship between different orders of fractional power series, thus showing that the basis elements corresponding to different orders n of fractional differential equations may intersect, i.e., solutions from a higher-order equation may inherit solutions from a lower-order equation under some initial conditions.

Let us consider two fractional Riccati equations (2.1.1)-(2.1.2) of orders p and q. Also, let $g = \gcd(p, q)$; $s^{(p)} = \frac{p}{g}$; $s^{(q)} = \frac{q}{g} \in \mathbb{N}$. Then, the solutions of these fractional Riccati equations will coincide if the following relations on their initial conditions hold true:

$$\begin{aligned} s_j^{(p)} &= 0, \qquad j \neq s^{(p)} l; \\ s_i^{(q)} &= 0, \qquad i \neq s^{(q)} l, \end{aligned} \tag{2.1.16}$$

where $l = 0, 1, \dots, n-1$.

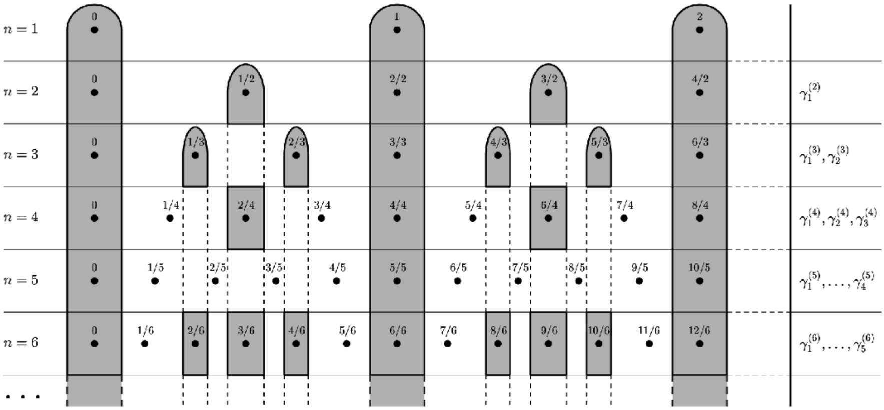


Fig. 2.1.4. Nested structure of Caputo fractional power series basis. Each row $\boldsymbol{n} = \boldsymbol{k};\ \boldsymbol{k} = \mathbf{1}, \mathbf{2}, \dots$ displays the powers of $\boldsymbol{x}$ in the Caputo fractional power series of the respective order. Parameters $\boldsymbol{\gamma}_{\boldsymbol{v}}^{(\boldsymbol{k})}; \boldsymbol{v} = \mathbf{1}, \mathbf{2}, \dots, \boldsymbol{k} - \mathbf{1}$ on the right depict the coefficients of the ODE (2.1.9) corresponding to the fractional initial conditions. Gray-filled sections correspond to the same power of $\boldsymbol{x}$ in respective base elements

Conclusions

In this paper, the concept of the Caputo fractional power series is used for the analysis of the fractional Riccati equation (2.1.1)-(2.1.2). It has been proven that the fractional Riccati equation (2.1.1)-(2.1.2) can be reduced to an integer-order ODE (2.1.9)-(2.1.10), which can be further investigated and solved via classical analytical or numerical techniques. Furthermore, it has been demonstrated via theoretical investigations, as well as computational experiments, that the solutions to fractional equations of different orders exhibit a nested structure: higher-order fractional Riccati equations inherit some solutions from lower-order equations when a subset of the initial conditions is set to zero.

2.2. *Review of The Extension of Analytic Solutions to FDEs to the Negative Half-Line*

2.2.1. Paper details

Title: The Extension of Analytic Solutions to FDEs to the Negative Half-Line

Authors: **Inga Timofejeva (now Telksnienė)**, Zenonas Navickas, Tadas Telksnys, Romas Marcinkevičius, Xiao-Jun Yang, and Minvydas Ragulskis

To be cited as: Timofejeva, Inga, et al. The Extension of Analytic Solutions to FDEs to the Negative Half-Line. *AIMS Mathematics 6.4* (2021): 3257–3271.

Input from I.Telksnienė: I. Telksnienė formalized and generalized the ideas related to the extension of solutions to CFDEs to the negative half-line, carried out most of the necessary computer algebra and numerical computations, and prepared the manuscript text.

2.2.2. Summary of the paper

Objective of the paper

This paper aims to:

1. Expand the methodology for the construction of fractional power series solutions, presented in the previous paper, to a more general class of equations – CFDEs with polynomial nonlinearity:
$$\left({}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)}\right)^n y(x) = \sum_{k=0}^{m} a_k y(x)^k; \quad m \in \mathbb{N}, a_m \neq 0, a_k \in \mathbb{C}.$$
2. Develop an analytical framework for the extension of solutions to such CFDEs to the negative half-line and investigate the properties of such an extension.

Methodology and results

The concepts of the Riemann extension algorithm for the Caputo functions and the generalized differential operator technique used in this paper are summarized before the introduction of main results to facilitate better understanding.

Preliminaries: Riemann extension algorithm for Caputo functions

The well-known idea that analytic functions can be extended beyond their radius of convergence [30] can be adapted for the Caputo fractional power series (1.4.1) as follows.

Let

$$f(x) = \sum_{j=0}^{+\infty} v_j w_j^{(n)} = \sum_{j=0}^{+\infty} \gamma_j \left(\sqrt[n]{x}\right)^j; \; v_j, \gamma_j \in \mathbb{C} \tag{2.2.1}$$

have the convergence radius $T_0 \in \mathbb{R}$ with respect to $\sqrt[n]{x}$.

Also, we choose x_0, such that $0 < x_0 < T_0^n$. Then (2.2.1) can be rearranged to:

$$f(x) = \sum_{j=0}^{+\infty} \hat{\gamma}_j \left(\sqrt[n]{x} - \sqrt[n]{x_0}\right)^j, \tag{2.2.2}$$

where coefficients $\hat{\gamma}_j$ are defined as:

$$\hat{\gamma}_j = \sum_{k=j}^{+\infty} \binom{k}{j} \gamma_k \left(\sqrt[n]{x_0}\right)^{k-j} \tag{2.2.3}$$

and are finite, since x_0 is in the convergence radius of function f. Let the convergence radius for (2.2.2) with respect to $\sqrt[n]{x} - \sqrt[n]{x_0}$ be $T_1 \in \mathbb{R}, T_1 > 0$. Moreover, if we choose $x_1 \neq x_0$, $0 < x_1 < T_0^n$ and repeat the procedure, then:

$$\sum_{j=0}^{+\infty} \hat{\gamma}_j \left(\sqrt[n]{x} - \sqrt[n]{x_0}\right)^j = \sum_{j=0}^{+\infty} \tilde{\gamma}_j \left(\sqrt[n]{x} - \sqrt[n]{x_1}\right)^j, \tag{2.2.4}$$

for such x with which both sides converge.

Thus, the above technique can be used to extend the convergence area of a function expressible as the Caputo fractional power series by rewriting it in a different basis $\left(\sqrt[n]{x} - \sqrt[n]{x_k}\right)^j, k = 0,1, \ldots$

We note that the Caputo differentiation operator $(1.4.9)$ is defined only for the basis $\left(\sqrt[n]{x}\right)^j$ in the neighborhood of $x = 0$; therefore, all the computations will be first executed in this neighborhood, and then extended to the entire function domain by using the technique presented above.

Preliminaries: Generalized differential operator technique

A brief description of the generalized differential operator technique [31; 32], used in this paper for the construction of the series solutions to ODEs, is given below.

Consider the following ODE:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = P(t,z); \;\; z(c) = s; \;\; c, s \in \mathbb{R}, \tag{2.2.5}$$

where P is an arbitrary analytic function. The generalized differential operator can be defined for (2.2.5) as follows [33]:

$$\mathbf{D}_{cs} = \mathbf{D}_c + P(c,s)\mathbf{D}_s, \tag{2.2.6}$$

where $\mathbf{D}_\lambda$ is the partial differentiation operator with respect to λ.

Then, solution to $(2.2.5)$ can be written in the series form as follows [31]:

$$z(t,c,s) = \sum_{j=0}^{+\infty} \frac{(t-c)^j}{j!} p_j(c,s) = \sum_{j=0}^{+\infty} \frac{(t-c)^j}{j!} \mathbf{D}_{cs}^j s\,. \tag{2.2.7}$$

Novel methodology and results

The following Caputo fractional differential equation is considered:

$$\left({}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)}\right)^n y_n = Q_m(y_n), \tag{2.2.8}$$

where $y_n = y_n(x) = \sum_{j=0}^{+\infty} v_j w_j^{(n)} \in {}^{C}\mathbb{F}_n$, and Q_m is an arbitrary m-th order polynomial:

$$Q_m(y_n) = \sum_{k=0}^{m} a_k y_n^k; \;\; m \in \mathbb{N}, a_m \neq 0, a_k \in \mathbb{C}. \tag{2.2.9}$$

Firstly, by using the techniques similar to the ones presented in the previous paper, it is shown that (2.2.8) can be reduced to the following ODE:

$$\frac{\mathrm{d}\hat{y}_n}{\mathrm{d}t} = n\left(t^{n-1}Q_m(\hat{y}_n) + \sum_{j=1}^{n-1} \frac{v_j}{\Gamma\left(\frac{j}{n}+1\right)} t^{j-1}\right), \tag{2.2.10}$$

where $y_n(x) = \hat{y}_n\left(\sqrt[n]{x}\right)$.

The application of the generalized differential operator technique, described above, as well as the relation $y_n(x) = \hat{y}_n\left(\sqrt[n]{x}\right)$ yields the following theorem:

Theorem 2.2.1. Consider the following Cauchy initial value problem:

$$\left({}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)}\right)^n y_n = Q_m(y_n), \quad (2.2.11)$$

$$y_n(x_0) = u_0; \;\; x_0 \in \mathbb{R}, x_0 \geq 0, \quad (2.2.12)$$

$$\left({}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)}\right)^k y_n\Bigg|_{x=0} = v_k; \;\; k = 1, \dots n-1. \quad (2.2.13)$$

CFDE (2.2.11)-(2.2.13) has the following fractional power series solution:

$$y_n(x; x_0, u_0, v_1, \dots v_{n-1}) = \sum_{j=0}^{+\infty} \frac{\left(\sqrt[n]{x} - \sqrt[n]{x_0}\right)^j}{j!} p_j\left(\sqrt[n]{x_0}, u_0\right), \quad (2.2.14),$$

where

$$p_j(c, s) = \mathbf{D}_{cs}^j s = \left(\mathbf{D}_c + n\left(c^{n-1}Q_m(s) + \sum_{j=1}^{n-1} \frac{v_j}{\Gamma\left(\frac{j}{n}+1\right)} c^{j-1}\right)\mathbf{D}_s\right)^j s, \quad (2.2.15)$$

if x satisfies $\left|\sqrt[n]{x} - \sqrt[n]{x_0}\right| < T_{x_0}$, where $T_{x_0} > 0$ is the convergence radius of (2.2.14).

End of theorem

We note that the obtained solution (2.2.14) can be extended by using the Riemann extension algorithm for Caputo functions, discussed above, by following these steps:

1. Choose a sequence $x_1, x_2, \dots$, such that $0 < x_0 < x_1 < x_2 < \cdots$.
2. Compute $u_1, u_2, \dots$ as:
$$u_{r+1} = y_n(x_{r+1}; x_r, u_r, v_1, \dots v_{n-1}), \qquad r = 0,1, \dots$$
3. Solution to (2.2.11)-(2.2.13) can be written as follows for any r:
$$y_n(x; x_0, u_0, v_1, \dots v_{n-1}) = y_n(x; x_r, u_r, v_1, \dots v_{n-1}),$$

for $\left|\sqrt[n]{x} - \sqrt[n]{x_r}\right| < T_{x_r}$, $T_{x_r} > 0$.

We also note that the aforementioned results can be extended to analytic functions by taking $Q_m(y_n) = \sum_{k=0}^{+\infty} a_k x^k$.

In order to enhance the understanding and application of the CFDEs, the feasibility of the extension of the solutions to CFDE (2.2.11)-(2.2.13) to the negative values of argument x was further explored. We note that such a possibility has not been reported previously since such extension is not possible when considering the original integral-based definition of the Caputo fractional derivative (1.2.2), as the integral is defined only for the non-negative values of x. If, however, the Caputo fractional differentiation operator is defined via the concept of a fractional power series (see Section 1.4), then the extension of solutions to the negative half-line is possible. It results in complex-valued fractional power series which shall be defined below.

Let us consider the following extension of the fractional power series basis functions (1.4.2) (see Section 1.4):

$$\left(w_j^{(n)}\right)_k = \frac{\left(\sqrt[n]{|x|}\right)^j}{\Gamma\left(\frac{j}{n}+1\right)} \exp\left(ij\frac{arg(x)+2\pi k}{n}\right), \quad (2.2.16)$$

where $x \in \mathbb{R}$; $k = 0,1, \dots n-1; j = 0,1, \dots$; $\sqrt[n]{|x|}$ is the real root and i is the complex unit. We note that the basis obtained with $k = 0$ coincides with the basis defined in Section 1.4 (for $x \geq 0$), while $\left(w_j^{(n)}\right)_k$ for $k = 1,2, \dots, n-1$ are complex-valued functions. Then, the Caputo fractional power series (1.4.1) can be extended into n complex-valued series as follows:

$$f_k(x) = \sum_{j=0}^{+\infty} v_j \left(w_j^{(n)}\right)_k; \; k = 0, \dots n-1. \quad (2.2.17)$$

When using (2.2.16) and (2.2.17), solution (2.2.14) to CFDE (2.2.11)-(2.2.13) can be extended into a complex plane as follows:

$$\left(y_n(x; x_0, u_0, v_1, \dots v_{n-1})\right)_k =$$

$$= \sum_{j=0}^{+\infty} \frac{\left(\sqrt[n]{|x|} - \sqrt[n]{|x_0|}\right)^j}{j!} \left(\lambda_j^{(k)}\left(|x_0|, \alpha, u_0^{(0)}\right) + i\mu_j^{(k)}\left(|x_0|, \alpha, u_0^{(0)}\right)\right), \quad (2.2.18)$$

where

$$\lambda_j^{(k)}\left(|x_0|, \alpha, u_0^{(0)}\right) = \mathrm{Re}\left(\left(\beta_n^{(k)}(\alpha)\right)^j p_j\left(\sqrt[n]{|x_0|}\beta_n^{(k)}(\alpha), u_0\right)\right); \quad (2.2.19)$$

$$\mu_j^{(k)}\left(|x_0|, \alpha, u_0^{(0)}\right) = \mathrm{Im}\left(\left(\beta_n^{(k)}(\alpha)\right)^j p_j\left(\sqrt[n]{|x_0|}\beta_n^{(k)}(\alpha), u_0\right)\right); \quad (2.2.20)$$

$$\alpha = \arg(x_0); \; \beta_n^{(k)}(\alpha) = \exp\left(i\frac{\alpha+2\pi k}{n}\right). \quad (2.2.21)$$

Expression (2.2.18) allows to consider solutions to (2.2.11)-(2.2.13) for $x < 0$, which are complex and multi-valued (n solutions corresponding to the number of unique roots to $\sqrt[n]{x}$).

In order to extend a particular solution from (2.2.18), corresponding to kth branch of root $\sqrt[n]{x}$, to the whole real line, the procedure described earlier (based on the Riemann extension algorithm for Caputo functions) is modified as follows:

1. Choose two sequences ... $< x_{-r} < \cdots < x_{-1} < 0$ and $0 < x_1 < x_2 < \cdots$
2. Compute two sequences $u_{-r}^{(k)}$ and $u_r^{(k)}$ as:

$$u_{r+1}^{(k)} = \left(y_n\left(x_{r+1}; x_r, u_r^{(k)}, v_1, \dots v_{n-1}\right)\right)_k, \qquad r = 0,1,\dots$$

$$u_{-r-1}^{(k)} = \left(y_n\left(x_{-r-1}; x_{-r}, u_{-r}^{(k)}, v_1, \dots v_{n-1}\right)\right)_k, \qquad r = 0,1,\dots$$

3. The solution to (2.2.11)-(2.2.13) corresponding to kth branch of root $\sqrt[n]{x}$ reads as follows:

$$\left(y_n(x)\right)_k = \left(y_n\left(x; x_r, u_r^{(k)}, v_1, \dots v_{n-1}\right)\right)_k, \qquad r = 0, \pm 1, \pm 2, \dots$$

We note that the sequences $\dots < x_{-r} < \dots < x_{-1} < 0$ and $0 < x_1 < x_2 < \dots$ have to be chosen in such a way that the resulting series would be convergent in $\mathbb{R}$ and $\mathbb{C}$. We also note that the functions $\left(y_n(x)\right)_k$ are non-differentiable at $x = 0$, since it is a branching point for the solution.

Computational experiments illustrating the proposed techniques are provided by using the following fractional Riccati equation as an example:

$$\left({}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)}\right)^n y = 2y^2 - 5y - 3. \tag{2.2.22}$$

Let $n = 2, x_0 = 0$ and

$$y(0) = u_0;\ {}^{C}\mathbf{D}_x^{\left(\frac{1}{2}\right)} y\Big|_{x=0} = v_1;\ u_0, v_1 \in \mathbb{R}. \tag{2.2.23}$$

The solutions to the resulting initial value problem are constructed via the technique outlined above and depicted in Fig. 2.2.1. We note that two solutions exist, since $\sqrt{x}$ has two branches. The solutions corresponding to two different values of v_1 are presented in Fig. 2.2.2. It can be seen that, as v_1 approaches zero, the solutions to (2.2.22) at $n = 2$ approach the solution to (2.2.22) at $n = 1$, i.e., the solution to the fractional CFDE approaches the solution to the ODE (the classical Riccati equation), which coincides with the results presented in the earlier paper.

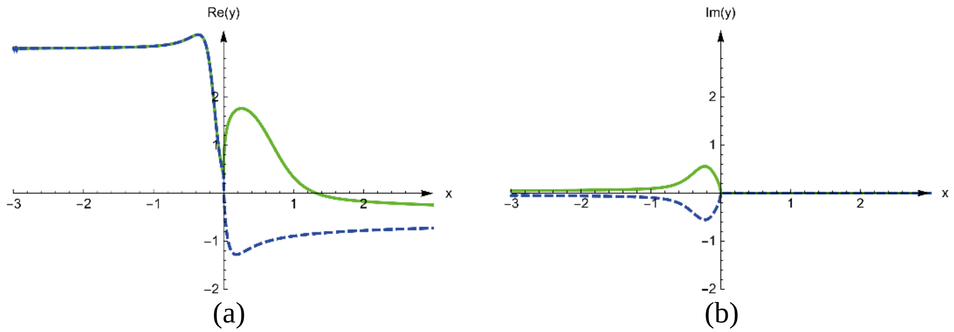
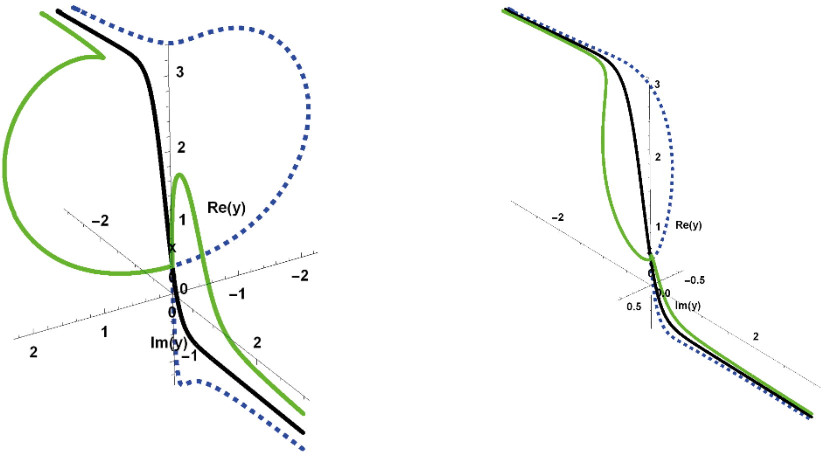


Fig. 2.2.1. Solutions to (2.2.22)-(2.2.23) for $\boldsymbol{n = 2}$, $\boldsymbol{u_0 = \frac{2}{5}}$, $\boldsymbol{v_1 = 5}$. Real and imaginary parts of the solutions are depicted in parts (a) and (b), respectively. Solid green and dashed blue lines denote solutions corresponding to different branches of $\boldsymbol{\sqrt{x}}$



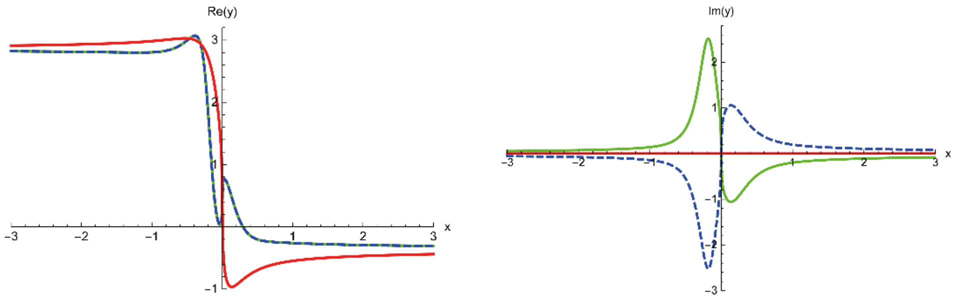
(a) (b)

Fig. 2.2.2. Solutions to (2.2.22)-(2.2.23) for $\boldsymbol{n = 2}$, $\boldsymbol{u_0 = \frac{2}{5}}$. Parts (a) and (b) correspond to initial conditions $\boldsymbol{v_1 = 5}$ and $\boldsymbol{v_1 = 1}$, respectively. Solid green and dashed blue lines denote solutions corresponding to different branches of $\sqrt{\boldsymbol{x}}$. Black line depicts the solution to (2.2.22) at $\boldsymbol{n = 1}$, i.e., the solution to the classical Riccati ODE

Let $n = 3, x_0 = 0$ and

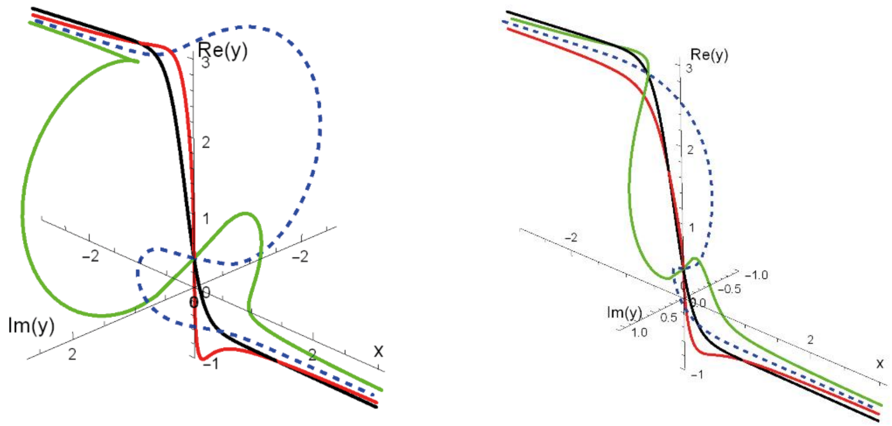
$$y(0) = u_0;\ \ {}^{C}\mathbf{D}_x^{\left(\frac{1}{3}\right)} y\bigg|_{x=0} = v_1;\ \ \left({}^{C}\mathbf{D}_x^{\left(\frac{1}{3}\right)}\right)^2 y\bigg|_{x=0} = v_2;\quad u_0, v_1, v_2 \in \mathbb{R}. \qquad (2.2.24)$$

The solutions to the resulting initial value problem are displayed in Fig. 2.2.3. In this case, there are three solutions: one real-valued, and two complex-valued. The solutions, corresponding to different values of v_1 and v_2, are depicted in Fig. 2.2.4.



(a) (b)

Fig. 2.2.3. Solutions to (2.2.22)-(2.2.23) for $\boldsymbol{n = 3}$, $\boldsymbol{u_0 = \frac{2}{5}}$, $\boldsymbol{v_1 = -1}$, $\boldsymbol{v_2 = \frac{1}{10}}$. The real and imaginary parts of the solutions are depicted in parts (a) and (b), respectively. Dashed blue and solid green and red lines denote solutions corresponding to different branches of $\sqrt[3]{\boldsymbol{x}}$



(a) (b)

Fig. 2.2.4. Solutions to (2.2.22)-(2.2.23) for $\boldsymbol{n} = \mathbf{3}$, $\boldsymbol{u_0} = \frac{\mathbf{2}}{\mathbf{5}}$. Parts (a) and (b) correspond to initial conditions $\boldsymbol{v_1} = -\mathbf{1}$, $\boldsymbol{v_2} = \frac{\mathbf{1}}{\mathbf{10}}$ and $\boldsymbol{v_1} = -\frac{\mathbf{1}}{\mathbf{2}}$, $\boldsymbol{v_2} = \frac{\mathbf{1}}{\mathbf{2}}$, respectively. Dashed blue and solid green and red lines denote solutions corresponding to different branches of $\sqrt[3]{\boldsymbol{x}}$. Black line depicts the solution to (2.2.22) at $\boldsymbol{n} = \mathbf{1}$, i.e., the solution to the classical Riccati ODE

Conclusions

In this paper, the concept of the Caputo fractional power series is used for the analysis of Caputo fractional differential equations with polynomial nonlinearity (2.2.8). It has been proven that such CFDE can be reduced to an integer-order ODE (2.2.10). The derived ODE can then be used to obtain the solution to the original CFDE in the form of fractional power series (2.2.14). The resulting solution can be extended by using Riemann extension techniques adapted to fractional power series to facilitate the analysis of the solution at a neighborhood different than the origin $x = 0$.

Moreover, an extension of the concept of fractional power series to the negative half-line has been presented. Such an extension can be applied to extend the solution of the CFDE to the entire real line, including the negative values of x. This possibility has not been reported previously. As demonstrated by the theoretical derivations as well as computational experiments, such a solution to CFDE is complex and multi-valued (n solutions, where n is is the denominator of the fractional differentiation order) for negative values of x.

2.3. Review of *An Operator-Based Scheme for the Numerical Integration of FDEs*

2.3.1. Paper details

Title: An Operator-Based Scheme for the Numerical Integration of FDEs

Authors: **Inga Timofejeva (now Telksnienė)**, Zenonas Navickas, Tadas Telksnys, Romas Marcinkevičius, and Minvydas Ragulskis

To be cited as: Timofejeva, Inga, et al. An Operator-Based Scheme for the Numerical Integration of FDEs. *Mathematics 9.12* (2021): 1372.

Input from I.Telksnienė: I. Telksnienė formalized the preliminary version of the operator-based scheme for the numerical integration of FDEs, performed most of the symbolic and numerical calculations, and was responsible for writing the manuscript text.

2.3.2. Summary of the paper

Objective of the paper

This paper aims to utilize the techniques established in the previous paper to develop a semi-analytical scheme for the construction of approximate solutions to CFDEs.

Methodology and results

The techniques, presented in the previous paper, are combined to create a preliminary framework of the semi-analytical scheme for the construction of approximate solutions to CFDEs of the following type:

$$\left({}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)}\right)^n y = Q_m(y), \tag{2.3.1}$$

$$y(x_0) = u_0; \;\; x_0 \in \mathbb{R}, x_0 \geq 0, \tag{2.3.2}$$

$$\left({}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)}\right)^k y\Bigg|_{x=0} = v_k; \;\; k = 1, \dots n-1, \tag{2.3.3}$$

where $Q_m(y) = \sum_{k=0}^{+\infty} a_k x^k$ is an arbitrary analytic function.

The steps of the semi-analytical integration scheme are as follows:

1. Select values of the following parameters: the order of the approximation N, the upper bound of the independent variable L.
2. Transform CFDE (2.3.1)-(2.3.3) into the characteristic ODE by using the procedure outlined in the previous paper:

$$\frac{\mathrm{d}\hat{y}}{\mathrm{d}x} = P(\hat{y}, v_1, \dots, v_{n-1}); \tag{2.3.4}$$

$$\hat{y}(c_0) = s_0, \tag{2.3.5}$$

where $c_0 = \sqrt[n]{x_0}, s_0 = u_0$.

3. Compute the analytic expressions of coefficients $p_j(c,s)$ $(j=0,\dots N)$ in the approximate truncated series solution (the exact solution is obtained if the series goes to $+\infty$)

$$\hat{y}_N(x,c,s)=\sum_{j=0}^{N}\frac{(x-c)^j}{j!}p_j(c,s) \qquad (2.3.6)$$

to ODE (2.3.4)-(2.3.5) as follows:

$$p_j(c,s)=\mathbf{D}_{cs}^{j}s=$$
$$=\left(\mathbf{D}_c+n\left(c^{n-1}Q_m(s)+\sum_{j=1}^{n-1}\frac{v_j}{\Gamma\left(\frac{j}{n}+1\right)}c^{j-1}\right)\mathbf{D}_s\right)^j s. \qquad (2.3.7)$$

4. Repeat the following steps until the upper bound L is reached $(k=0,1,\dots)$:
   - Evaluate coefficients $p_j(c_k,s_k),\ j=0,1,\dots,N$.
   - Find the lowest value of x at which a pre-selected error tolerance criteria is violated.
   - Compute the new initial values:

     $$c_{k+1}=x-\varepsilon;\ \ s_{k+1}=\hat{y}_N(c_{k+1},c_k,s_k), \qquad (2.3.8)$$

     where ε is an arbitrary small number.
5. Merge the obtained segments of the numerical solution to the ODE (2.3.4)-(2.3.5) to form the piecewise-polynomial approximation $\hat{y}_N(x)$:

$$\hat{y}_N(x)=\hat{y}_N(x,c_k,s_k),\ \ c_k\le x<c_{k+1},\quad k=0,1,\dots \qquad (2.3.9)$$

6. Construct the semi-analytical approximation of the solution to the CFDE (2.3.1)-(2.3.3) by applying $y_N(x)=\hat{y}_N\left(\sqrt[n]{x}\right)$.

Naturally, in order to apply the scheme described above, it is necessary to devise a step-size $h_k=c_k-c_{k-1}$ selection strategy which would ensure a desired level of computational errors between the exact and the estimated solutions. For that purpose, the following numerical investigation is performed:

a) CFDE with a known analytic closed-form solution is selected:

$$x\left({}^{C}\mathbf{D}_x^{\left(\frac{1}{2}\right)}\right)^2 y=1-2y+y^2; \qquad (2.3.10)$$

$$y(1)=1;\ \ {}^{C}\mathbf{D}_x^{\left(\frac{1}{2}\right)}y\Big|_{x=0}=-1. \qquad (2.3.11)$$

It can be shown (by using the technique presented in the previous paper) that (2.3.10)-(2.3.11) can be reduced to ODE:

$$\frac{\mathrm{d}\hat{y}}{\mathrm{d}x}=\frac{2(1-2\hat{y}+\hat{y}^2)}{x}-\frac{2}{\sqrt{\pi}}; \qquad (2.3.12)$$

$$\hat{y}(1)=1;\ \hat{y}=\hat{y}(x);\ \hat{y}\left(\sqrt{x}\right)=y(x), \qquad (2.3.13)$$

which has the following analytic closed-form solution [1]:

$$\hat{y}(x) = \frac{\gamma_1\left(Y_1(\gamma_1)J_1(\gamma_2) - J_1(\gamma_1)Y_1(\gamma_2)\right)}{4\left(Y_0(\gamma_1)J_1(\gamma_2) - J_0(\gamma_1)Y_1(\gamma_2)\right)} + 1, \qquad (2.3.14)$$

where $\gamma_1 = 4\sqrt{-\frac{x}{\sqrt{\pi}}}$, $\gamma_2 = 4\sqrt{-\frac{1}{\sqrt{\pi}}}$; $J_\beta(x)$ and $Y_\beta(x)$ are Bessel functions of the first and second kind, respectively. Despite the fact that γ_1, γ_2 are complex, the solution $\hat{y}(x)$ is real.

b) Parameters values $N = 6$ and $\delta = 10^{-5}$ are selected for the numerical investigation, performed below, where N is the order of the approximation, and δ is the maximal allowed level of computational errors.

c) Piecewise-polynomial approximation $\hat{y}_N(x)$ of (2.3.12)-(2.3.13) is computed by using the scheme presented above. The step-size of x in the fourth step of the procedure is selected in such a way that the absolute errors between the approximate solution and the exact solution (2.3.14) would not exceed δ. The final piecewise-polynomial approximation $\hat{y}_N(x)$ (2.3.12)-(2.3.13) is depicted in Fig. 2.3.1.

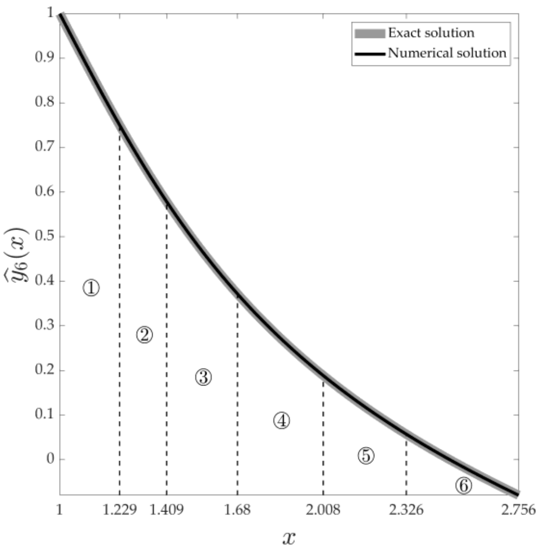


Fig. 2.3.1. Grey and black solid lines correspond to the exact solution (2.3.14) and the piecewise-polynomial approximation to (2.3.12)-(2.3.13), respectively ($\boldsymbol{N = 6}$, $\boldsymbol{\delta = 10^{-5}}$). The parts of the approximate solution obtained at different steps are separated by black dashed lines. Circled digits denote the step number

d) The relation between the step-size h_k and the change in the approximate solution $\hat{y}_N(x)$ at each step is approximated via the following linear regression equation:

$$\Delta\hat{y}_N = \kappa_0^{(N)} + \kappa_1^{(N)}h. \qquad (2.3.15)$$

For this instance ($N = 6$), the equation reads: $\Delta\hat{y}_N = 0.26572 - 0.29293h$.

The empirical results obtained in the course of this numerical investigation can be incorporated into the semi-analytical integration scheme, presented earlier for the adaptive selection of the step-size. For example, the step “Find the lowest value of x at which the pre-selected error tolerance criterion is violated” could be reformulated as:
“Find the lowest value of x at which at least one of the following conditions is violated:

$$h_k(x) = x - c_k \le h^{(U)}; \tag{2.3.16}$$

$$\Delta\hat{y}_N^{(k)}(x) = \max_{c_k \le \tilde{x} \le x} \hat{y}_N(\tilde{x}, c_k, s_k) - \max_{c_k \le \tilde{x} \le x} \hat{y}_N(\tilde{x}, c_k, s_k) \le \Delta\hat{y}_N^{(U)}; \tag{2.3.17}$$

$$\Delta\hat{y}_N^{(k)}(x) \le \kappa_0^{(N)} + \kappa_1^{(N)} h_k(x), \tag{2.3.18}$$

where $h^{(U)}$ and $\Delta\hat{y}_N^{(U)}$ are the upper bounds for the step-size and the change in the numerical solution, which can be derived from the previous numerical investigation as the highest values of h and $\Delta\hat{y}_N$ on the regression line.”

The semi-analytical scheme for the construction of approximate solutions to CFDEs together with the technique for the adaptive selection of the step-size, as presented earlier, are then applied to the following CFDE:

$$x\left({}^{C}\mathbf{D}_x^{\left(\frac{1}{2}\right)}\right)^2 y = 1 - 2y + y^2 - y^3; \tag{2.3.19}$$

$$y(1) = 1;\ \ {}^{C}\mathbf{D}_x^{\left(\frac{1}{2}\right)} y\Big|_{x=0} = -1. \tag{2.3.20}$$

We note that the exact solution to (2.3.19)-(2.3.20) cannot be expressed in a closed form. The results of the applied techniques are displayed in Fig. 2.3.2.

It is important to note that the technique for the selection of the step-size, as outlined above, serves as an example and would need considerable refinement for the application in the general case since the parameters $h^{(U)}$, $\Delta\hat{y}_N^{(U)}$, $\kappa_0^{(N)}$, $\kappa_1^{(N)}$ are based on only one empirical experiment with respect to a specific CFDE (2.3.10)-(2.3.11). A development of a more efficient and robust step-size selection strategy as well as a comparison of the proposed framework with the already established semi-analytical and numerical techniques for the integration of CFDEs remains a definite objective of future research.

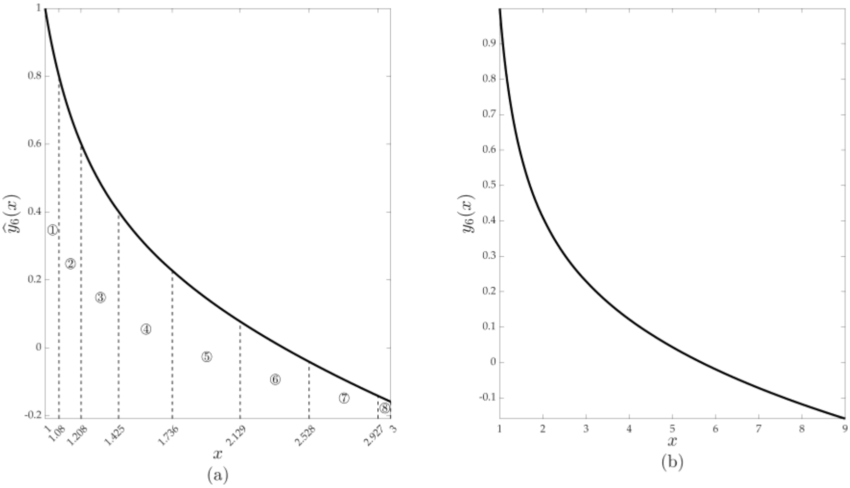


Fig. 2.3.2. Application of the semi-analytical FDE integration scheme to (2.3.19)-(2.3.20). Part (a) depicts the numerical solution to the characteristic ODE ($\boldsymbol{N} = \boldsymbol{6}, \boldsymbol{L} = \boldsymbol{3}, \boldsymbol{\delta} = \boldsymbol{10^{-5}}$). Parts of the approximate solution obtained at different steps are separated by black dashed lines. Circled digits denote the step number. Part (b) displays the piecewise-polynomial approximation to the initial FDE (2.3.19)-(2.3.20)

Conclusions

In this paper, the concepts and techniques developed in the previous study have been utilized in order to develop a preliminary framework of the semi-analytical scheme for the construction of piecewise-polynomial approximate solutions to CFDEs of the type (2.3.1)-(2.3.3). A numerical investigation using a CFDE with a known analytic closed-form solution is performed to analyze the relation between the order of approximation, its accuracy, the change in the approximate solution, and the step-size of the algorithm. Further theoretical and empirical findings from the investigation of these relations could potentially be used to develop a robust and efficient technique for the adaptive selection of the step-size, which could be incorporated into the presented integration scheme. Other directions in future studies could also explore a modification of the scheme so that to allow the use of any numerical integration method during the solution of the characteristic ODE. Although this alteration would transform the scheme into a purely numerical one and pose challenges in adjusting the timescale (since the approximation would no longer be a polynomial function), it could open up new possibilities in applying the already available results.

2.4. Review of *The Construction of Solutions to* ${}^{C}D^{(1/n)}$ *Type FDEs via Reduction to* $\left({}^{C}D^{\left(\frac{1}{n}\right)}\right)^{n}$ *Type FDEs*

2.4.1. Paper details

Title: The Construction of Solutions to ${}^{C}\mathbf{D}^{(1/n)}$ Type FDEs via Reduction to $\left({}^{C}\mathbf{D}^{\left(\frac{1}{n}\right)}\right)^{n}$ Type FDEs

Authors: Romas Marcinkevičius, **Inga Telksnienė**, Tadas Telksnys, Zenonas Navickas, and Minvydas Ragulskis

To be cited as: Marcinkevicius, Romas, et al. The Construction of Solutions to ${}^{C}\mathbf{D}^{(1/n)}$ Type FDEs via Reduction to $\left({}^{C}\mathbf{D}^{\left(\frac{1}{n}\right)}\right)^{n}$ Type FDEs. *AIMS Mathematics 7.9* (2022): 16536–16554, doi: 10.3934/math.2022905.

Input from I.Telksnienė: I. Telksnienė formalized and generalized the ideas of Z. Navickas and T. Telksnys, assisted R. Marcinkevičius with computer algebra computations, prepared all the necessary computational examples, drafted and refined the manuscript text, and corresponded with the editorial office.

2.4.2. Summary of the paper

Objective of the paper

This paper aims to expand the methodologies for the construction of fractional power series solutions, presented in the previous papers, to a more general class of equations, specifically, ${}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)}$ type CFDEs:

$${}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)}y(x) = G(x, y),$$

where $G(x, y)$ is an analytic function.

Methodology and results

In the previous studies, the focus has been on the analysis of the $\left({}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)}\right)^{n}$ type CFDEs:

$$\left({}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)}\right)^{n} y(x) = F(x, y), \tag{2.4.1}$$

where $F(x, y)$ is an analytic function. While these investigations have yielded substantial theoretical insights, their applicability in practical scenarios is quite limited. This study seeks to bridge this gap by illustrating that the methodologies previously developed for the $\left({}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)}\right)^n$ type CFDEs can be utilized for the construction of solutions to the more general and applicable ${}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)}$ type CFDEs:

$$ {}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)} y(x) = G(x,y), \qquad (2.4.2), $$

where $G(x,y)$ is an analytic function.

It has been demonstrated in this paper that (2.4.2) can be transformed into (2.4.1) if specific conditions hold true, which can then be solved by applying the previously developed techniques.

Without the loss of generality and for the sake of simplicity, the order of the Caputo derivative is set to $\alpha = \frac{1}{2}$ for the remainder of this paper. Nevertheless, the findings outlined here can be readily generalized to any order $\alpha = \frac{1}{n}, n \in \mathbb{N}$.

For the clarity of presentation, this paper focuses on applying the described scheme to Riccati-type FDEs. Nonetheless, similar analytical and numerical computations can be performed for a general CFDE of type (2.4.2).

The following theorem is outlined and proven in this paper.

Theorem 2.4.1. Consider the following two Cauchy problems:

First CFDE:

$$ {}^{C}\mathbf{D}_x^{\left(\frac{1}{2}\right)} y_1 = a_2 y_1^2 + a_1 y_1 + a_0 + \Phi(x); \qquad (2.4.3) $$
$$ y_1(0) = \gamma_0, $$

where $a_2, a_1, a_0, \gamma_0 \in \mathbb{R}$, and $\Phi(x)$ is an arbitrary fractional power series:

$$ \Phi(x) = \sum_{j=1}^{+\infty} \phi_j w_j^{(2)} \in {}^{C}\mathbb{F}_2, \qquad \phi_j \in \mathbb{R}. \qquad (2.4.4) $$

Second CFDE:

$$ \left({}^{C}\mathbf{D}_x^{\left(\frac{1}{2}\right)}\right)^2 y_2 = b_3 y_2^3 + b_2 y_2^2 + b_1 y_2 + \Psi(x); $$
$$ y_2(0) = \lambda_0; \quad {}^{C}\mathbf{D}_x^{\left(\frac{1}{2}\right)} y_2 \Big|_{x=0} = \lambda_1, \qquad (2.4.5) $$

where $b_3, b_2, b_1, \lambda_0, \lambda_1 \in \mathbb{R}$, and $\Psi(x)$ is an arbitrary fractional power series:

$$ \Psi(x) = \sum_{j=0}^{+\infty} \psi_j w_j^{(2)} \in {}^{C}\mathbb{F}_2, \qquad \psi_j \in \mathbb{R}. \qquad (2.4.6) $$

Cauchy problems (2.4.3) and (2.4.5) have the same solution $y_1 = y_2 = y = \sum_{j=0}^{+\infty} \gamma_j w_j^{(2)}$ if the following relations hold true:

$$ \Psi(x) = {}^{C}\mathbf{D}_x^{\left(\frac{1}{2}\right)} \Phi(x) + a_2 \Theta_{y_1}(x) + 2a_2 y_1 \Phi(x) + a_1\big(a_0 + \Phi(x)\big); \qquad (2.4.7) $$
$$ b_3 = 2a_2^2; $$
$$ b_2 = 3a_1 a_2; \qquad (2.4.8) $$
$$ b_1 = a_1^2 + 2a_0 a_2; $$

$$\lambda_0 = \gamma_0; \quad \lambda_1 = a_2\gamma_0^2 + a_1\gamma_0 + a_0, \tag{2.4.9}$$

where

$$\Theta_{y_1}(x) = \sum_{j=0}^{+\infty} \theta_j w_j^{(2)}; \;\; \theta_0 = 0,$$

$$\theta_j = \sum_{k=1}^{j} \frac{1}{\Gamma\left(\frac{k}{2}+1\right)} \left(\frac{\Gamma\left(\frac{j+3}{2}\right)}{\Gamma\left(\frac{j-k+3}{2}\right)} - 2\frac{\Gamma\left(\frac{j}{2}+1\right)}{\Gamma\left(\frac{j-k}{2}+1\right)} \right) \gamma_k \gamma_{j-k+1}, \quad j = 1,2, \ldots.$$

End of theorem

Theorem 2.4.1 allows to use techniques already developed in previous papers to solve (2.4.2) type CFDEs via the scheme displayed in Fig. 2.4.1.

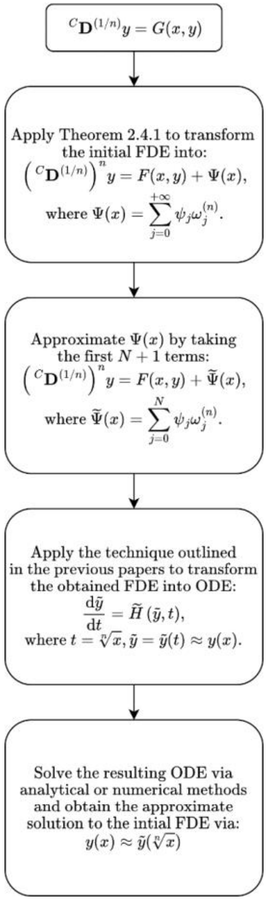


Fig. 2.4.1. Schematic diagram of the algorithm for transforming ${}^C\mathbf{D}_x^{\left(\frac{1}{n}\right)}$ type CFDEs into $\left({}^C\mathbf{D}_x^{\left(\frac{1}{n}\right)}\right)^n$ type CFDEs

The presented scheme (Fig. 2.4.1) is then applied to the following Cauchy problem for Riccati CFDE:

$$
\begin{gathered}
{}^{C}\mathbf{D}_x^{\left(\frac{1}{2}\right)} y = \frac{1}{4}y^2 + \frac{1}{2}y - \frac{1}{3}; \\
y(0) = \frac{1}{10}.
\end{gathered} \tag{2.4.10}
$$

By using Theorem 2.4.1, (2.4.10) can be transformed into the following Cauchy problem:

$$
\begin{gathered}
\left({}^{C}\mathbf{D}_x^{\left(\frac{1}{2}\right)}\right)^2 y = \frac{1}{8}y^3 + \frac{3}{8}y^2 + \frac{1}{12}y + \Psi(x); \\
y(0) = \frac{1}{10}; \quad {}^{C}\mathbf{D}_x^{\left(\frac{1}{2}\right)} y\Big|_{x=0} = -\frac{337}{1200},
\end{gathered} \tag{2.4.11}
$$

where the coefficients of $\Psi(x) = \sum_{j=0}^{+\infty} \psi_j w_j^{(2)}$ are obtained via (2.4.7).

Next, by using the techniques, presented in the previous papers, (2.4.11) can be converted into the following ODE:

$$
\frac{\mathrm{d}\hat{y}}{\mathrm{d}t} = 2t\left(\frac{1}{8}\hat{y}^3 + \frac{3}{8}\hat{y}^2 + \frac{1}{12}\hat{y} + \Psi(t^2)\right) - \frac{337}{1200\Gamma\left(\frac{3}{2}\right)}; \quad \hat{y}(0) = \frac{1}{10}, \tag{2.4.12}
$$

where $t = \sqrt{x}$ and $\hat{y} = \hat{y}(t) = y(x)$. We note that the function $\Psi(t^2)$ is represented by an infinite power series with no known closed form. Thus, ODE (2.4.12) cannot be solved directly – $\Psi(t^2)$ should first be approximated, for example, by taking the first $N + 1$ terms:

$$
\begin{gathered}
\frac{\mathrm{d}\tilde{y}}{\mathrm{d}t} = 2t\left(\frac{1}{8}\tilde{y}^3 + \frac{3}{8}\tilde{y}^2 + \frac{1}{12}\tilde{y} + \sum_{j=0}^{N} \psi_j \frac{t^{2j}}{\Gamma\left(1 + \frac{j}{n}\right)}\right) - \frac{337}{1200\Gamma\left(\frac{3}{2}\right)}; \\
\tilde{y}(0) = \frac{1}{10},
\end{gathered} \tag{2.4.13}
$$

where $\tilde{y}$ tends to $\hat{y}$ as N tends to infinity.

The approximate solution to (2.4.10) (semi-analytical or numerical, depending on the method) can now be obtained by solving (2.4.13) and applying $y(x) \approx \tilde{y}\left(\sqrt{x}\right)$. Fig. 2.4.2 displays the solutions to (2.4.13) and (2.4.10), respectively, for different values of N. These solutions are compared with a numerical solution to (2.4.10) computed via the direct Garrappa's method [34]. It can be observed that that the approximate solution obtained via the method presented in this paper approaches the Garrappa's solution as N increases.

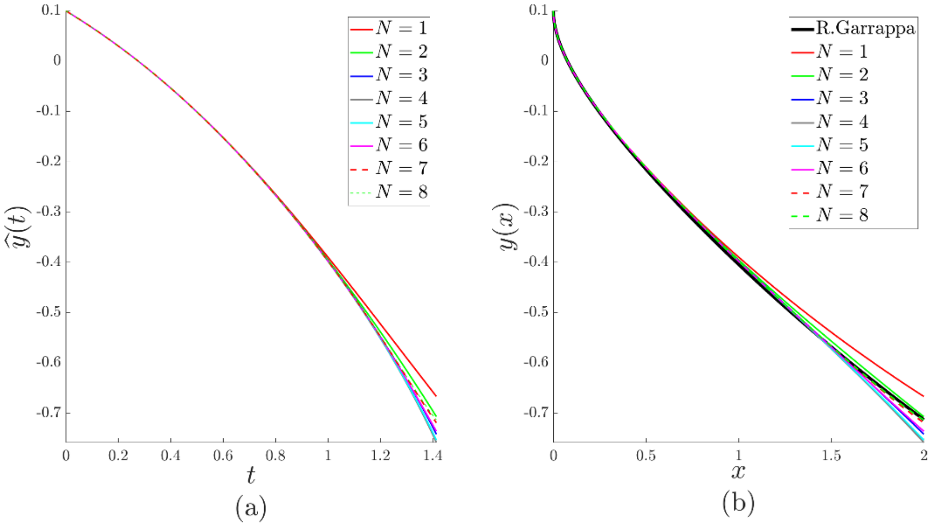


Fig. 2.4.2. Part (a) displays approximate solutions to ODE $(\mathbf{2.4.13})$ for different values of $\boldsymbol{N}$. Part (b) depicts approximate solutions to CFDE $(\mathbf{2.4.10})$ for different values of $\boldsymbol{N}$ as well as a numerical solution to $(\mathbf{2.4.10})$ computed via Garrappa's method [34] (black solid line)

Conclusions

In this paper, the concept of the Caputo fractional power series is used for the analysis and solution of ${}^{C}\mathbf{D}_{x}^{\left(\frac{1}{n}\right)}$ Caputo fractional differential equations (2.4.2). It has been shown that (2.4.2) type CFDEs can be transformed into $\left({}^{C}\mathbf{D}^{\left(\frac{1}{n}\right)}\right)^{n}$ if specific conditions hold true, which can then be solved by applying the techniques developed in the previous papers. A computational example has been provided to demonstrate the application of the proposed technique.

3. CONCLUSIONS

A completely novel approach for the analysis of Caputo fractional differential equations (CFDE) has been developed in the Thesis. This approach is based on the concept of the Caputo fractional power series and the algebraic realization of the Caputo fractional differentiation operator. Essentially, the presented methodology is a generalization of operator calculus from ordinary to fractional differential equations, thereby providing a completely fresh perspective on the construction of both numerical and analytical solutions to CFDEs. The application of the proposed approach to various types of CFDEs has yielded the following conclusions:

1. Solutions to CFDEs of different orders exhibit a nested structure: higher-order Caputo fractional equations inherit some solutions from lower-order equations when a subset of initial conditions is set to zero.
2. The Caputo fractional differential equation with polynomial nonlinearity (2.2.8) can be reduced to an integer-order ODE (2.2.10) by using the developed methodology.
3. Solutions to the CFDEs can be extended to the negative half-line via the methodology presented in Section 2.2.2. Such solutions are complex and multi-valued for negative values of x.
4. The developed approach, based on the concept of the Caputo fractional power series, can be utilized for the construction of approximate solutions to the CFDEs, which results in a novel straightforward adaptive semi-analytical scheme.
5. Type ${}^{C}\mathbf{D}_{x}^{\left(\frac{1}{n}\right)}$ CFDEs can be transformed into type $\left({}^{C}\mathbf{D}^{\left(\frac{1}{n}\right)}\right)^{n}$ CFDEs if specific conditions (listed in Theorem 2.4.1) hold true, which can then be solved via the techniques presented in the Thesis.

4. SANTRAUKA

4.1. Įvadas

Diferencialinės lygtys – viena pamatinių matematikos šakų, pritaikoma daugelyje mokslo sričių (fizikoje, inžinerijoje, biologijoje, ekonomikoje ir kt.) modeliuojant bei analizuojant įvairius reiškinius.

Pastaraisiais metais vis didesnis dėmesys skiriamas specifinei diferencialinių lygčių klasei – trupmeninės eilės diferencialinėms lygtims (TDL). Skirtingai nuo klasikinių diferencialinių lygčių, kuriose naudojamos sveikosios eilės išvestinės, TDL yra naudojamos trupmeninės eilės išvestinės, praplečiančios klasikinių diferencijavimo bei integravimo operatorių eilę iki realiųjų arba kompleksinių skaičių.

Nuo trupmeninės išvestinės sąvokos atsiradimo XVII a. [10] iki šių dienų buvo suformuluota daugiau kaip dvidešimt apibrėžimų, realizuojančių trupmenines išvestines, kurių kiekvienas pasižymi unikaliomis savybėmis ir specifinėmis taikymo sritimis. Didžiausiu pritaikomumu modeliavimo srityje pasižymintiems trupmeninės išvestinės apibrėžimams būdinga nelokalumo savybė, t. y. priešingai nei klasikinės išvestinės, trupmeninės išvestinės reikšmė taške priklauso ne tik nuo funkcijos reikšmių to taško aplinkoje, bet ir nuo visų praėjusių funkcijos reikšmių. Ši savybė suteikia galimybę modeliuoti sistemas, pasižyminčias atminties arba paveldimumo efektais, kai dabartinė sistemos būsena priklauso nuo visų praeities būsenų.

Šiame darbe nagrinėjamos trupmeninės eilės diferencialinės lygtys, formuluojamos naudojant vieną populiariausių trupmeninės eilės išvestinių – italų mokslininko M. Caputo pasiūlytą apibrėžimą [12]. Caputo trupmeninės eilės diferencialinės lygtys (CTDL) itin palankios modeliavimui ir sėkmingai taikomos fizikos (pvz., plazmos fizikos [12], optikos [13]), inžinerijos (pvz., viskoelastingumo [14]), biologijos (pvz., gamtos sistemų [15], aplinkos inžinerijos [16]), ekonomikos ir finansų (pvz., konkurencijos modelių [17]), sveikatos mokslų (pvz., epidemiologijos [18], neurobiologijos [19]) ir kt. srityse.

Dėl plataus Caputo diferencialinių lygčių taikymo spektro šių lygčių tyrimas įvairiais metodais, tiek analitiniais, tiek skaitiniais, itin aktualus uždavinys. Taigi, pagrindinis šios **disertacijos tikslas** – sukurti naują pusiau analitinę schemą, skirtą Caputo trupmeninių diferencialinių lygčių sprendiniams konstruoti bei analizuoti, panaudojant Caputo trupmeninių laipsninių eilučių algebros koncepciją, pateiktą [1; 2].

Šiam tikslui įgyvendinti buvo iškelti penki uždaviniai:

1. Naujos metodologijos, skirtos $\left({}^{C}\mathrm{D}^{\left(\frac{1}{n}\right)}\right)^{n}$ tipo Rikati CTDL sprendiniams konstruoti, sukūrimas bei sprendinių, išreikštų trupmeninėmis laipsninėmis eilutėmis, struktūros tyrimas.
2. Metodologijos, skirtos CDTL sprendiniams konstruoti, išplėtimas, taikytinas platesnei klasei lygčių – $\left({}^{C}\mathrm{D}^{\left(\frac{1}{n}\right)}\right)^{n}$ tipo CTDL su daugianario tipo netiesiškumu.

3. Analitinės metodologijos, leidžiančios išplėsti CTDL sprendinius į neigiamąją realiąją pusašę, sukūrimas ir šio plėtinio savybių tyrimas.
4. Pusiau analitinės schemos, skirtos apytiksliams CTDL sprendiniams konstruoti, sukūrimas.
5. Metodologijos, skirtos CDTL sprendiniams konstruoti, išplėtimas, taikytinas dar platesnės klasės netiesinėms lygtims – ${}^{C}\mathrm{D}_{x}^{\left(\frac{1}{n}\right)}$ tipo CDTL.

Ši daktaro disertacija yra parengta mokslinių straipsnių rinkinio pagrindu, kur kiekvienas straipsnis atitinka vieno ar kelių aukščiau aprašytų uždavinių sprendimą. Pirmasis straipsnis „The fractal structure of analytical solutions to fractional Riccati equation“ atvėrė duris šiam tyrimui, kadangi jame buvo sukurta nauja metodologija, skirta specifinės Rikati tipo CTDL sprendiniams konstruoti. Pristatyti rezultatai buvo reikšmingai išplėtoti darbe „The extension of analytic solutions to FDEs to the negative half-line“, parodant, kad patobulinta metodologija gali būti panaudota ne tik Rikati tipo CTDL, bet ir platesnei klasei lygčių, pavyzdžiui, CTDL su daugianario tipo netiesiškumu. Kitas žingsnis, kurio rezultatai paskelbti straipsnyje „An operator-based scheme for the numerical integration of FDEs“, buvo skirtas sukurtos metodikos pritaikymui kuriant pusiau analitinę schemą, skirtą apytiksliams CTDL sprendiniams konstruoti. Paskutiniame disertaciją sudarančiame straipsnyje „The construction of solutions to ${}^{C}\mathbf{D}^{(1/n)}$ type FDEs via reduction to $\left({}^{C}\mathbf{D}^{\left(\frac{1}{n}\right)}\right)^{n}$ type FDEs“, sujungus visus ankstesnių tyrimų rezultatus, buvo sukurtas naujas metodas, skirtas dar platesnės klasės netiesinių CTDL sprendiniams konstruoti.

4.2. Įvadas į Caputo trupmeninių laipsninių eilučių algebrą

Kadangi disertacijoje yra naudojama Caputo trupmeninių laipsninių eilučių algebros koncepcija [1; 2], šiame skyriuje pateikiamos pagrindinės sąvokos, suteikiančios kontekstą disertaciją sudarančioms publikacijoms.

Pažymėkime Caputo trupmeninės išvestinės eilę $\alpha = \frac{k}{n}$, čia $k, n \in \mathbb{N}$ ir $\gcd(k, n) = 1$. Taip pat tarkime, kad $x \geq 0$.

Toliau šiame darbe nagrinėsime funkcijas, išreikštas Caputo trupmeninėmis laipsninėmis eilutėmis, t. y. laipsninėmis eilutėmis, kurių elementai turi trupmeninius laipsnius:

$$f(x) = \sum_{j=0}^{+\infty} v_j w_j^{(n)}, \tag{4.2.1}$$

čia $v_j \in \mathbb{R}$ yra eilutės koeficientai, o $w_j^{(n)}, n \in \mathbb{N}, j = 0,1, \ldots$ yra trupmeninės laipsninės eilutės n-tosios eilės bazinės funkcijos, apibrėžiamos tokiu būdu:

$$w_j^{(n)} = \frac{x^{\frac{j}{n}}}{\Gamma\left(\frac{j}{n} + 1\right)}, j = 0,1, \ldots \tag{4.2.2}$$

Caputo trupmeninių laipsninių eilučių aibė, atitinkanti parametrą n, žymima:

$$ {}^{C}\mathbb{F}_n = \left\{ \sum_{j=0}^{+\infty} v_j w_j^{(n)}; \quad c_j \in \mathbb{C} \right\}. \tag{4.2.3} $$

Aibė ${}^{C}\mathbb{F}_n$ su standartinėmis sudėties, daugybos iš skaliaro ir sandaugos operacijomis sudaro algebrą virš $\mathbb{C}$. Ši algebra vadinama Caputo algebra ir žymima:

$$ {}^{C}\mathcal{F}_n = \langle {}^{C}\mathbb{F}_n; +, \cdot \,|\mathbb{C} \rangle \tag{4.2.4} $$

Caputo $\frac{1}{n}$-osios eilės trupmeninis diferencijavimo operatorius bazinėms funkcijoms $w_j^{(n)}$ yra apibrėžiamas tokiu būdu:

$$ {}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)} w_j^{(n)} = \begin{cases} 0, & j = 0 \\ w_{j-1}^{(n)}, & j = 1,2,\ldots \end{cases} \tag{4.2.5} $$

Tuomet funkcijos $f(x) = \sum_{j=0}^{+\infty} v_j w_j^{(n)} \in {}^{C}\mathbb{F}_n$ Caputo $\alpha = \frac{k}{n}$-osios eilės trupmeninė išvestinė:

$$ {}^{C}\mathbf{D}_x^{\left(\frac{k}{n}\right)} f(x) = \left({}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)} \right)^k f(x) = \sum_{j=0}^{+\infty} v_{j+k} w_j^{(n)} \in {}^{C}\mathbb{F}_n. \tag{4.2.6} $$

Svarbu pastebėti, kad šis Caputo trupmeninio diferencijavimo operatoriaus apibrėžimas sutampa su originaliu integraliniu apibrėžimu [2].

4.3. Svarbiausi darbo rezultatai

Kadangi ši disertacija yra ginama mokslinių straipsnių rinkinio pagrindu, tolimesniuose poskyriuose pateikiamos disertaciją sudarančių publikacijų santraukos.

4.3.1. Straipsnio „The fractal structure of analytical solutions to fractional Riccati equation“ rezultatų santrauka

Straipsnio tikslas

Šiame straipsnyje:

1. pristatyta nauja metodologija, skirta žemiau pateiktos Rikati tipo Caputo trupmeninės diferencialinės lygties sprendiniams konstruoti:

$$ \left({}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)} \right)^n y(x) = a_2 y(x)^2 + a_1 y(x) + a_0; \;\; a_0, a_1, a_2 \in \mathbb{C}; \; n \in \mathbb{N}, $$

2. ištirta Rikati tipo CTDL sprendinių, išreikštų trupmeninėmis laipsninėmis eilutėmis, struktūra.

Metodika ir rezultatai

Straipsnyje nagrinėjamas šis Rikati tipo CTDL Koši uždavinys:

$$ \left({}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)} \right)^n y_n = a_2 y_n^2 + a_1 y_n + a_0; \tag{4.3.1} $$

$$\left({}^{C}\mathbf{D}_{x}^{\left(\frac{1}{n}\right)} \right)^{k} y_n \Bigg|_{x=0} = s_k^{(n)}; k = 0, \dots n-1, \tag{4.3.2}$$

čia $a_0, a_1, a_2 \in \mathbb{C}$, $n \in \mathbb{N}$, $y_n = y_n\left(x; s_0^{(n)}, s_1^{(n)}, \dots, s_{n-1}^{(n)}\right) \in {}^{C}\mathbb{F}_n$, o parametrai $s_0^{(n)}, s_1^{(n)}, \dots, s_{n-1}^{(n)}$ žymi pradines sąlygas, atitinkančias $x = 0$.

Svarbu pastebėti, kad kai $n > 1$, operatorius $\left({}^{C}\mathbf{D}_{x}^{\left(\frac{1}{n}\right)} \right)^{n}$ nėra tapatus $\frac{\mathrm{d}}{\mathrm{d}x}$, kadangi pirmasis operatorius veikia Caputo trupmeninę laipsninę eilutę (4.2.1), sudarytą iš trupmeninių x laipsnių, o operatorius $\frac{\mathrm{d}}{\mathrm{d}x}$ taikomas klasikinėms Teiloro laipsninėms eilutėms, sudarytoms tik iš sveikųjų x laipsnių.

Pažymėkime:

$$y_n = \sum_{j=0}^{+\infty} c_j w_j^{(n)} = \sum_{j=0}^{+\infty} \gamma_j^{(n)} \Gamma\left(\frac{j}{n} + 1\right) w_j^{(n)} = \sum_{j=0}^{+\infty} \gamma_j^{(n)} x^{\frac{j}{n}};\ c_j \in \mathbb{C}. \tag{4.3.3}$$

Pirmiausia išvedami nagrinėjamos lygties sprendinio koeficientų rekurentiniai sąryšiai. Tai įvykdoma įterpiant (4.3.3) į (4.3.1)-(4.3.2) ir atliekant algebrinius pertvarkymus su gautomis išraiškomis. Gaunami šie sąryšiai:

$$\gamma_k^{(n)} = \frac{s_k^{(n)}}{\Gamma\left(\frac{k}{n} + 1\right)}; k = 0,1, \dots, n-1; \tag{4.3.4}$$

$$(j+n)\gamma_{j+n}^{(n)} = n\left(a_2 \sum_{r=0}^{j} \left(\gamma_r^{(n)} \gamma_{j-r}^{(n)} \right) + a_1 \gamma_j^{(n)} + \delta_j a_0 \right); j = 0,1, \dots, \tag{4.3.5}$$

čia $\delta_j = 1$ jei $j = 0$ ir $\delta_j = 0$ priešingu atveju.

Toliau apibrėžiama sekos $\left(\gamma_j^{(n)}; j = 0,1, \dots\right)$ charakteringoji funkcija:

$$\varphi_n(t) = \sum_{j=0}^{+\infty} \gamma_j^{(n)} t^j. \tag{4.3.6}$$

Rekurentinį sąryšį (4.3.5) galima pertvarkyti į šią paprastąją diferencialinę lygtį (charakteringosios funkcijos $\varphi_n(t)$ atžvilgiu):

$$\frac{\mathrm{d}\varphi_n}{\mathrm{d}t} = n t^{n-1}\left(a_2 \varphi_n^2(t) + a_1 \varphi_n(t) + a_0\right) + \sum_{j=1}^{n-1} j \gamma_j^{(n)} t^{j-1}. \tag{4.3.7}$$

Funkcija $\varphi_n(t)$ gali būti naudojama nagrinėjamos lygties (4.3.1)–(4.3.2) sprendiniams išreikšti, kadangi:

$$y_n = \sum_{j=0}^{+\infty} \gamma_j^{(n)} x^{\frac{j}{n}} = \varphi_n\left(\sqrt[n]{x}\right). \tag{4.3.8}$$

Taigi, Rikati trupmeninės diferencialinės lygties Koši uždavinys (4.3.1)–(4.3.2) yra ekvivalentus šiam paprastosios diferencialinės lygties (PDL) Koši uždaviniui:

$$\frac{\mathrm{d}}{\mathrm{d}x}(y_n - \psi_n) = a_2 y_n^2 + a_1 y_n + a_0; \qquad (4.3.9)$$

$$y_n(0) = s_0^{(n)} = \gamma_0^{(n)}, \qquad (4.3.10)$$

čia $\psi_n(x) = \sum_{j=0}^{n-1} \gamma_j^{(n)} x^{\frac{j}{n}}$.

Žinoma, gautą PDL galima analizuoti ir integruoti bet kuriais klasikiniais analitiniais ar skaitiniais metodais, taigi tai suteikia galimybę tirti šio tipo CTDL nekuriant naujų metodų.

Ištyrus sąryšių (4.3.5) ir (4.3.9) struktūrą galima pastebėti, kad jei lygties (4.3.1)–(4.3.2) pradinės sąlygos tenkina $s_1^{(n)} = s_2^{(n)} = \cdots = s_{n-1}^{(n)} = 0$, tuomet $\psi_n(x) = 0$ ir $\gamma_j^{(n)} \neq 0$ tik tada, jei $j = kn, k \in \mathbb{N}$, o tai reiškia, kad tokiu atveju sprendinys y_n priklauso aibei ${}^{C}\mathbb{F}_1$ ir tenkina klasikinę paprastąją Rikati lygtį:

$$\frac{\mathrm{d}y_n}{\mathrm{d}x} = a_2 y_n^2 + a_1 y_n + a_0. \qquad (4.3.11)$$

Taigi, bet kokios eilės $n \in \mathbb{N}$ trupmeninei Rikati lygčiai (4.3.1) tinka visi PDL (4.3.11) sprendiniai (jei sąryšis $s_1^{(n)} = s_2^{(n)} = \cdots = s_{n-1}^{(n)} = 0$ galioja), bet ji turi ir daugiau unikalių sprendinių. Straipsnyje taip pat pateikiami skaitiniai eksperimentai, patvirtinantys aukščiau pateiktus analitinius rezultatus.

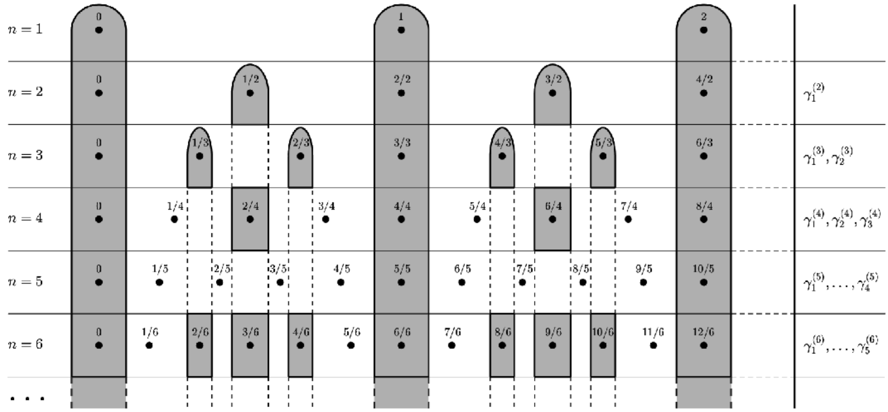
Minėtus pastebėjimus apie trupmeninės Rikati lygties sprendinių paveldimumą galima išplėsti: bet kuri $n = km$; $k, m \in \mathbb{N}$ eilės trupmeninė Rikati lygtis (4.3.1) paveldi sprendinius iš Rikati lygčių, kurių eilės $n = k$ ir $n = m$.

4.3.1 pav. pavaizduotas ryšys tarp skirtingų trupmeninių laipsninių eilučių eilių, kuriame matome, kad baziniai elementai, atitinkantys skirtingos eilės trupmeninių diferencialinių lygčių sprendinius, gali susikirsti, t. y., esant tam tikroms pradinėms sąlygoms, aukštesnės eilės lygties sprendiniai gali paveldėti žemesnės eilės lygties sprendinius.

Panagrinėkime dvi trupmenines Rikati lygtis (4.3.1)-(4.3.2), kurių eilės p ir q. Pažymėkime $g = \gcd(p, q)$; $s^{(p)} = \frac{p}{g}$; $s^{(q)} = \frac{q}{g} \in \mathbb{N}$. Tuomet šių trupmeninių Rikati lygčių sprendiniai sutampa, jei galioja šie pradinių sąlygų sąryšiai:

$$\begin{aligned} s_j^{(p)} &= 0, \qquad j \neq s^{(p)} l; \\ s_i^{(q)} &= 0, \qquad i \neq s^{(q)} l, \end{aligned} \qquad (4.3.12)$$

čia $l = 0, 1, \dots, n-1$.



4.3.1 pav. Caputo trupmeninių laipsninių eilučių bazių struktūra. Kiekvienoje eilutėje $\boldsymbol{n = k}$; $\boldsymbol{k = 1, 2, \dots}$ surašyti atitinkamos eilės Caputo trupmeninės laipsninės eilutės $\boldsymbol{x}$ laipsniai. Parametrai $\boldsymbol{\gamma_v^{(k)}}$; $\boldsymbol{v = 1, 2, \dots, k-1}$ dešinėje žymi PDL (4.3.9) koeficientus, atitinkančius trupmenines pradines sąlygas. Pilkai nuspalvintos dalys atitinka tuos pačius $\boldsymbol{x}$ laipsnius skirtingų eilių baziniuose elementuose

Išvados

Šiame straipsnyje Caputo trupmeninių laipsninių eilučių koncepcija panaudota trupmeninės Rikati lygties (4.3.1)–(4.3.2) analizei. Įrodyta, kad trupmeninė Rikati lygtis (4.3.1)–(4.3.2) gali būti redukuota į sveikosios eilės paprastąją diferencialinę lygtį (4.3.9)–(4.3.10), kurią galima toliau tirti ir spręsti klasikiniais analitiniais arba skaitiniais metodais. Be to, teorinių bei skaitinių tyrimų metu pademonstruota, kad skirtingų eilių trupmeninių lygčių sprendiniai pasižymi paveldimumu: aukštesnės eilės trupmeninės Rikati lygtys paveldi tam tikrus sprendinius iš žemesnės eilės lygčių, kai pradinių sąlygų poaibis yra lygus nuliui.

4.3.2. Straipsnio „The extension of analytic solutions to FDEs to the negative half-line“ rezultatų santrauka

Straipsnio tikslas

Šiame straipsnyje:

1. išplėsta ankstesnėje publikacijoje pateikta metodologija, skirta trupmeninių diferencialinių lygčių sprendiniams konstruoti. Patobulinta metodologija gali būti taikoma platesnei klasei lygčių – CTDL su daugianario tipo netiesiškumu:
$$\left({}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)}\right)^n y(x) = \sum_{k=0}^{m} a_k y(x)^k; \quad m \in \mathbb{N}, a_m \neq 0, a_k \in \mathbb{C}.$$
2. sukurta analitinė metodologija, leidžianti išplėsti šių CTDL sprendinius į neigiamąją realiąją pusašę ir ištirtos tokio plėtinio savybės.

Metodika ir rezultatai

Straipsnyje nagrinėjama tokio tipo Caputo trupmeninė diferencialinė lygtis:

$$\left({}^{C}\mathbf{D}_{x}^{\left(\frac{1}{n}\right)} \right)^{n} y_n = Q_m(y_n), \tag{4.3.13}$$

čia $y_n = y_n(x) = \sum_{j=0}^{+\infty} v_j w_j^{(n)} \in {}^{C}\mathbb{F}_n$ ir Q_m bet koks m-tosios eilės daugianaris:

$$Q_m(y_n) = \sum_{k=0}^{m} a_k y_n^k;\ \ m \in \mathbb{N}, a_m \neq 0, a_k \in \mathbb{C}. \tag{4.3.14}$$

Pirmiausia, taikant pertvarkymus, analogiškus pateiktiems ankstesniame straipsnyje, įrodoma, kad (4.3.13) galima redukuoti į tokią paprastąją diferencialinę lygtį:

$$\frac{\mathrm{d}\hat{y}_n}{\mathrm{d}t} = n\left(t^{n-1} Q_m(\hat{y}_n) + \sum_{j=1}^{n-1} \frac{v_j}{\Gamma\left(\frac{j}{n}+1\right)} t^{j-1} \right), \tag{4.3.15}$$

čia $y_n(x) = \hat{y}_n\left(\sqrt[n]{x}\right)$.

Tuomet, pritaikius apibendrinto diferencialinio operatoriaus metodą, aprašytą [33], bei sąryšį $y_n(x) = \hat{y}_n\left(\sqrt[n]{x}\right)$, įrodoma ši teorema:

Teorema 4.3.1. Nagrinėkime tokį Koši uždavinį:

$$\left({}^{C}\mathbf{D}_{x}^{\left(\frac{1}{n}\right)} \right)^{n} y_n = Q_m(y_n), \tag{4.3.16}$$

$$y_n(x_0) = u_0;\ \ x_0 \in \mathbb{R}, x_0 \geq 0, \tag{4.3.17}$$

$$\left({}^{C}\mathbf{D}_{x}^{\left(\frac{1}{n}\right)} \right)^{k} y_n \bigg|_{x=0} = v_k;\ \ k = 1, \dots n-1. \tag{4.3.18}$$

CTDL (4.3.16)–(4.3.18) turi šį trupmeninės laipsninės eilutės formos sprendinį:

$$y_n(x; x_0, u_0, v_1, \dots v_{n-1}) = \sum_{j=0}^{+\infty} \frac{\left(\sqrt[n]{x} - \sqrt[n]{x_0}\right)^j}{j!} p_j\left(\sqrt[n]{x_0}, u_0\right), \tag{4.3.19}$$

čia

$$p_j(c,s) = \mathbf{D}_{cs}^{j} s = \left(\mathbf{D}_c + n\left(c^{n-1} Q_m(s) + \sum_{j=1}^{n-1} \frac{v_j}{\Gamma\left(\frac{j}{n}+1\right)} c^{j-1} \right) \mathbf{D}_s \right)^{j} s, \tag{4.3.20}$$

jei x tenkina $\left|\sqrt[n]{x} - \sqrt[n]{x_0}\right| < T_{x_0}$, kur $T_{x_0} > 0$ yra (4.3.19) konvergavimo spindulys.

Teoremos pabaiga

Pastebėsime, kad gautą sprendinį (4.3.20) galima išplėsti adaptavus Rymano išplėtimo algoritmą Caputo trupmeninėms laipsninėms eilutėms. Taip pat pastebėsime, kad anksčiau pateiktus rezultatus galima išplėsti iki analizinių funkcijų $Q_m(y_n) = \sum_{k=0}^{+\infty} a_k x^k$.

Toliau pateikiama analitinė metodologija, leidžianti išplėsti CTDL (4.3.16)–(4.3.18) sprendinius neigiamoms argumento x reikšmėms. Svarbu pastebėti, kad toks išplėtimas negalimas nagrinėjant originalų integralinį Caputo trupmeninės išvestinės apibrėžimą [12], kadangi integralas apibrėžtas tik neneigiamoms reikšmėms x. Tačiau jei Caputo trupmeninis diferencijavimo operatorius apibrėžiamas naudojant trupmeninių laipsninių eilučių sąvoką (žr. 4.2 skyrių), tuomet sprendinių išplėtimas į neigiamąją realiąją pusašę yra įmanomas. Tokiu atveju gaunamos kompleksinės trupmeninės laipsninės eilutės, apibrėžtos žemiau.

Panagrinėkime tokį trupmeninių laipsninių eilučių bazinių funkcijų (4.2.2) plėtinį (žr. 4.2 skyrių):

$$\left(w_j^{(n)}\right)_k = \frac{\left(\sqrt[n]{|x|}\right)^j}{\Gamma\left(\frac{j}{n}+1\right)} \exp\left(ij\frac{\arg(x)+2\pi k}{n}\right), \tag{4.3.21}$$

čia $x \in \mathbb{R}$; $k = 0,1,\dots n-1; j = 0,1,\dots$; $\sqrt[n]{|x|}$ yra realioji šaknis, o i žymi menamąjį vienetą. Atkreipsime dėmesį, kad baziniai elementai, gauti esant $k = 0$ sutampa su 4.2 skyriuje apibrėžtais baziniais elementais (kai $x \geq 0$), o $\left(w_j^{(n)}\right)_k$, $k = 1,2,\dots,n-1$ yra kompleksinės funkcijos. Tuomet Caputo trupmeninę laipsninę eilutę (4.2.1) galima praplėsti iki n kompleksinių eilučių tokiu būdu:

$$f_k(x) = \sum_{j=0}^{+\infty} v_j \left(w_j^{(n)}\right)_k; \ k = 0,\dots n-1. \tag{4.3.22}$$

Panaudojus (4.3.21) ir (4.3.22), CTDL (4.3.16)–(4.3.18) sprendinį, galima išplėsti į kompleksinę plokštumą:

$$\left(y_n(x; x_0, u_0, v_1, \dots v_{n-1})\right)_k =$$
$$= \sum_{j=0}^{+\infty} \frac{\left(\sqrt[n]{|x|} - \sqrt[n]{|x_0|}\right)^j}{j!} \left(\lambda_j^{(k)}\left(|x_0|, \alpha, u_0^{(0)}\right) + i\mu_j^{(k)}\left(|x_0|, \alpha, u_0^{(0)}\right)\right), \tag{4.3.23}$$

kur

$$\lambda_j^{(k)}\left(|x_0|, \alpha, u_0^{(0)}\right) = \operatorname{Re}\left(\left(\beta_n^{(k)}(\alpha)\right)^j p_j\left(\sqrt[n]{|x_0|}\beta_n^{(k)}(\alpha), u_0\right)\right); \tag{4.3.24}$$

$$\mu_j^{(k)}\left(|x_0|, \alpha, u_0^{(0)}\right) = \operatorname{Im}\left(\left(\beta_n^{(k)}(\alpha)\right)^j p_j\left(\sqrt[n]{|x_0|}\beta_n^{(k)}(\alpha), u_0\right)\right); \tag{4.3.25}$$

$$\alpha = \arg(x_0); \ \beta_n^{(k)}(\alpha) = \exp\left(i\frac{\alpha+2\pi k}{n}\right). \tag{4.3.26}$$

Išraiška (4.3.23) leidžia nagrinėti CTDL (4.3.16)–(4.3.18) sprendinius, kai $x < 0$. Tokiu atveju gaunami sprendiniai yra daugiareikšmiai ir kompleksiniai (n sprendinių, atitinkančių unikalias skaičiaus $\sqrt[n]{x}$ šaknis).

Pastebėsime, kad gautą sprendinį (4.3.23), atitinkantį k-tąją šaknies $\sqrt[n]{x}$ šaką, galima išplėsti į visą realiąją ašį, adaptavus Rymano išplėtimo algoritmą Caputo praplėstoms trupmeninėms laipsninėms eilutėms su neigiamomis argumento reikšmėmis.

Išvados

Šiame straipsnyje Caputo trupmeninių laipsninių eilučių koncepcija buvo panaudota trupmeninių diferencialinių lygčių su polinominiu netiesiškumu (4.3.13) analizei. Įrodyta, kad tokią CTDL galima redukuoti į sveikosios eilės PDL (4.3.15). Tuomet gautoji PDL gali būti panaudota norint sukonstruoti pradinės CTDL sprendinį trupmeninės laipsninės eilutės (4.3.23) pavidalu. Gautas sprendinys gali būti išplėstas naudojant Rymano išplėtimo metodus, adaptuotus trupmeninėms laipsninėms eilutėms, kad būtų įmanoma analizuoti sprendinį kitoje negu $x = 0$ aplinkoje.

Taip pat pateiktas trupmeninių laipsninių eilučių sąvokos išplėtimas iki neigiamos realiosios pusašės. Toks išplėtimas gali būti taikomas CTDL sprendiniui išplėsti iki visos realiosios ašies, įskaitant neigiamas x reikšmes. Teoriniai bei skaitiniai eksperimentai rodo, kad tokiu atveju gaunami CTDL sprendiniai, atitinkantys neigiamas x reikšmes, yra daugiareikšmiai ir kompleksiniai (n sprendinių, kur n yra trupmeninės diferencijavimo eilės vardiklis).

4.3.3. Straipsnio „An operator-based scheme for the numerical integration of FDEs“ rezultatų santrauka

Straipsnio tikslas

Šiame straipsnyje anksčiau pristatytos metodologijos pritaikomos kuriant pusiau analitinę schemą, skirtą apytikslių CTDL sprendinių konstravimui.

Metodika ir rezultatai

Nagrinėkime tokio tipo CTDL:

$$\left({}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)}\right)^n y = Q_m(y), \tag{4.3.27}$$

$$y(x_0) = u_0; \;\; x_0 \in \mathbb{R}, x_0 \geq 0, \tag{4.3.28}$$

$$\left({}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)}\right)^k y\Bigg|_{x=0} = v_k; \;\; k = 1, \ldots n-1, \tag{4.3.29}$$

čia $Q_m(y) = \sum_{k=0}^{+\infty} a_k x^k$ yra bet kokia analizinė funkcija.

Siūlomos pusiau analitinės integravimo schemos etapai yra pateikiami žemiau:

1. Pasirenkame šių parametrų vertes: aproksimacijos eilė N, nepriklausomojo kintamojo viršutinis rėžis L.
2. Pertvarkome CTDL (4.3.27)-(4.3.29) į charakteringąją PDL, naudodami ankstesniame straipsnyje aprašytą procedūrą:

$$\frac{\mathrm{d}\hat{y}}{\mathrm{d}x} = P(\hat{y}, v_1, \ldots, v_{n-1}); \tag{4.3.30}$$

$$\hat{y}(c_0) = s_0, \tag{4.3.31}$$

čia $c_0 = \sqrt[n]{x_0}, s_0 = u_0$.

3. Apskaičiuojame TDL (4.3.30)-(4.3.31) apytikslio eilutės formos sprendinio (tikslus sprendinys gaunamas, jei eilutė tęsiasi iki $+\infty$)

$$\hat{y}_N(x, c, s) = \sum_{j=0}^{N} \frac{(x-c)^j}{j!} p_j(c, s) \tag{4.3.32}$$

koeficientų $p_j(c, s)$ $(j = 0, \dots N)$ analitines išraiškas:

$$p_j(c, s) = \mathbf{D}_{cs}^{j} s =$$

$$= \left(\mathbf{D}_c + n \left(c^{n-1} Q_m(s) + \sum_{j=1}^{n-1} \frac{v_j}{\Gamma\left(\frac{j}{n}+1\right)} c^{j-1} \right) \mathbf{D}_s \right)^j s. \tag{4.3.33}$$

4. Kartojame šiuos veiksmus, kol bus pasiektas viršutinis rėžis L $(k = 0,1, \dots)$:
   - Įvertiname koeficientus $p_j(c_k, s_k)$, $j = 0,1, \dots, N$.
   - Randame mažiausią x vertę, kuriai esant pažeidžiamas iš anksto pasirinktas leistinas paklaidos lygmuo.
   - Apskaičiuojame naujas pradines vertes:

     $$c_{k+1} = x - \varepsilon;\;\; s_{k+1} = \hat{y}_N(c_{k+1}, c_k, s_k), \tag{4.3.34}$$

     kur ε yra bet koks mažas skaičius.
5. Sujungiame gautus apytikslio TDL (4.3.30)–(4.3.31) sprendinio segmentus ir suformuojame dalimis-polinominę aproksimaciją $\hat{y}_N(x)$:

$$\hat{y}_N(x) = \hat{y}_N(x, c_k, s_k),\;\; c_k \le x < c_{k+1},\;\;\; k = 0,1, \dots \tag{4.3.35}$$

6. Sudarome pradinės CTDL (4.3.27)–(4.3.29) sprendinio pusiau analitinę aproksimaciją taikydami $y(x) = \hat{y}_N\left(\sqrt[n]{x}\right)$.

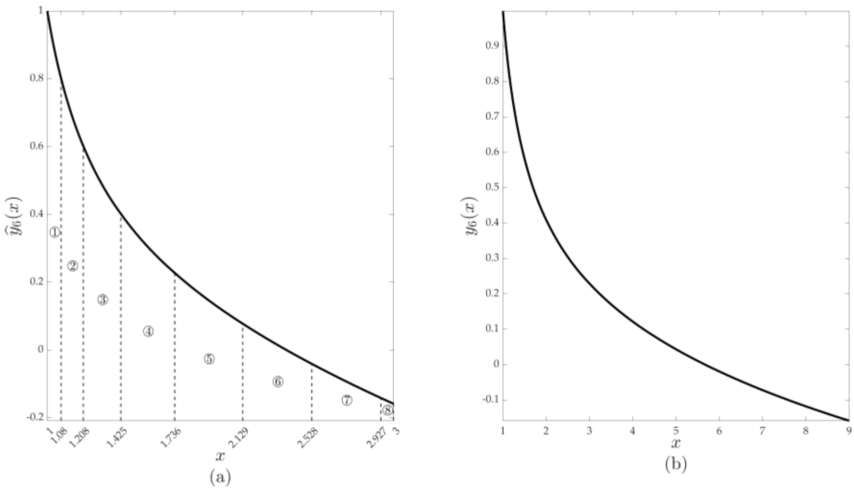
Norint taikyti aukščiau aprašytą schemą, būtina sukurti žingsnio $h_k = c_k - c_{k-1}$ dydžio parinkimo strategiją, kuri užtikrintų pageidaujamą skirtumo tarp tikslių ir apytikslių sprendinių lygį. Šiuo tikslu buvo atlikti keli skaitiniai tyrimai, kurių metu buvo analizuojama CTDL su žinomu analitiniu uždarosios formos sprendiniu. Šio skaitinio tyrimo metu gauti empiriniai rezultatai gali būti įtraukti į anksčiau pateiktą pusiau analitinę integravimo schemą adaptyviam žingsnio dydžio parinkimui.

Anksčiau pateikta pusiau analitinė CTDL apytikslių sprendinių sudarymo schema ir adaptyvaus žingsnio dydžio parinkimo metodas pritaikomi žemiau pateiktai CTDL:

$$x \left({}^{C}\mathbf{D}_x^{\left(\frac{1}{2}\right)} \right)^2 y = 1 - 2y + y^2 - y^3; \tag{4.3.36}$$

$$y(1) = 1;\;\; {}^{C}\mathbf{D}_x^{\left(\frac{1}{2}\right)} y \Big|_{x=0} = -1. \tag{4.3.37}$$

Aukščiau pristatytus metodus pritaikius šiai lygčiai, gauti rezultatai pateikti 4.3.2 pav.



4.3.2 pav. Pusiau analitinės integravimo schemos taikymas CTDL (4.3.36)–(4.3.37). Dalyje (a) pavaizduotas apytikslis charakteringosios PDL sprendinys ($\boldsymbol{N = 6, L = 3, \delta = 10^{-5}}$). Apytikslio sprendinio dalys, gautos skirtingų žingsnių metu, atskirtos juodomis punktyrinėmis linijomis. Apskritimuose esantys skaitmenys žymi žingsnio numerį. (b) dalyje pavaizduota pradinės CTDL (4.3.36)–(4.3.37) sprendinio dalimis-polinominė aproksimacija

Išvados

Šiame straipsnyje anksčiau pristatytos metodologijos pritaikytos kuriant preliminarią pusiau analitinę schemą, skirtą dalimis-polinominių apytikslių CTDL (4.3.27)–(4.3.29) sprendinių konstravimui. Siekiant išanalizuoti ryšius tarp aproksimacijos eilės, jos tikslumo, apytikslio sprendinio pokyčio bei algoritmo žingsnio dydžio, atliktas skaitinis tyrimas naudojant CTDL su žinomu analitiniu uždarosios formos sprendiniu. Tolesni teoriniai ir empiriniai šių ryšių tyrimo rezultatai galėtų būti panaudoti kuriant patikimą ir efektyvų adaptyvaus žingsnio dydžio parinkimo metodą, kuris galėtų būti įtrauktas į pristatytą integravimo schemą.

4.3.4. Straipsnio „The construction of solutions to ${}^{C}\mathbf{D}^{(1/n)}$ type FDEs via reduction to $\left({}^{C}\mathbf{D}^{\left(\frac{1}{n}\right)}\right)^{n}$ type FDEs“ rezultatų santrauka

Straipsnio tikslas

Šiame straipsnyje, sujungus visus ankstesnių tyrimų rezultatus, sukurtas naujas CTDL sprendinių konstravimo metodas, taikytinas dar platesnės klasės netiesinėms CTDL:

$$ {}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)} y(x) = G(x, y), $$

čia $G(x, y)$ yra analizinė funkcija.

Metodika ir rezultatai

Ankstesnėse publikacijose tirtos $\left({}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)}\right)^n$ tipo CTDL:

$$\left({}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)}\right)^n y(x) = F(x,y), \tag{4.3.38}$$

čia $F(x,y)$ – analizinė funkcija. Nors šių tyrimų metu gautos svarbios teorinės įžvalgos, jų praktinis pritaikomumas yra gana ribotas. Šiuo tyrimu siekiama užpildyti šią spragą, pademonstruojant, kad anksčiau sukurtos metodikos, skirtos $\left({}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)}\right)^n$ tipo CTDL, gali būti panaudotos didesniu praktiniu pritaikomumu pasižymintčioms ${}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)}$ tipo CDTL lygtims spręsti:

$$ {}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)} y(x) = G(x,y), \tag{4.3.39}$$

čia $G(x,y)$ yra analizinė funkcija.

Šiame straipsnyje įrodyta, kad, esant tam tikroms sąlygoms, CTDL (4.3.39) gali būti transformuota į (4.3.38), kuri savo ruožtu gali būti sprendžiama taikant anksčiau pristatytas metodologijas.

Nemažinant bendrumo ir vardan aiškumo skaitytojui, tolimesniems išvedimams pasirenkama Rikati tipo CTDL, kurios eilė $\alpha = \frac{1}{2}$. Nepaisant to, analogiški analitiniai bei skaitiniai pertvarkymai gali būti pritaikyti bet kokios eilės $\alpha = \frac{1}{n}, n \in \mathbb{N}$ CTDL (4.3.39).

Straipsnyje pateikiamas šios teoremos įrodymas.

Teorema 4.3.2. Nagrinėkime šiuos du Koši uždavinius:

Pirmoji CTDL:

$$ {}^{C}\mathbf{D}_x^{\left(\frac{1}{2}\right)} y_1 = a_2 y_1^2 + a_1 y_1 + a_0 + \Phi(x); \qquad y_1(0) = \gamma_0, \tag{4.3.40}$$

čia $a_2, a_1, a_0, \gamma_0 \in \mathbb{R}$ ir $\Phi(x)$ yra bet kokia trupmeninė laipsninė eilutė:

$$\Phi(x) = \sum_{j=1}^{+\infty} \phi_j w_j^{(2)} \in {}^{C}\mathbb{F}_2, \qquad \phi_j \in \mathbb{R}. \tag{4.3.41}$$

Antroji CTDL:

$$\left({}^{C}\mathbf{D}^{\left(\frac{1}{2}\right)}\right)^2 y_2 = b_3 y_2^3 + b_2 y_2^2 + b_1 y_2 + \Psi(x); \qquad y_2(0) = \lambda_0; \quad {}^{C}\mathbf{D}_x^{\left(\frac{1}{2}\right)} y_2 \Big|_{x=0} = \lambda_1, \tag{4.3.42}$$

čia $b_3, b_2, b_1, \lambda_0, \lambda_1 \in \mathbb{R}$ ir $\Psi(x)$ yra bet kokia trupmeninė laipsninė eilutė:

$$\Psi(x) = \sum_{j=0}^{+\infty} \psi_j w_j^{(2)} \in {}^{C}\mathbb{F}_2, \qquad \psi_j \in \mathbb{R}. \tag{4.3.43}$$

Koši uždaviniai (4.3.40) ir (4.3.42) turi tą patį sprendinį $y_1 = y_2 = y = \sum_{j=0}^{+\infty} \gamma_j w_j^{(2)}$, kai galioja šie sąryšiai:

$$\Psi(x) = {}^{C}\mathbf{D}_x^{\left(\frac{1}{2}\right)} \Phi(x) + a_2 \Theta_{y_1}(x) + 2a_2 y_1 \Phi(x) + a_1\big(a_0 + \Phi(x)\big); \qquad (4.3.44)$$

$$b_3 = 2a_2^2;$$

$$b_2 = 3a_1 a_2; \qquad (4.3.45)$$

$$b_1 = a_1^2 + 2a_0 a_2;$$

$$\lambda_0 = \gamma_0; \quad \lambda_1 = a_2\gamma_0^2 + a_1\gamma_0 + a_0, \qquad (4.3.46)$$

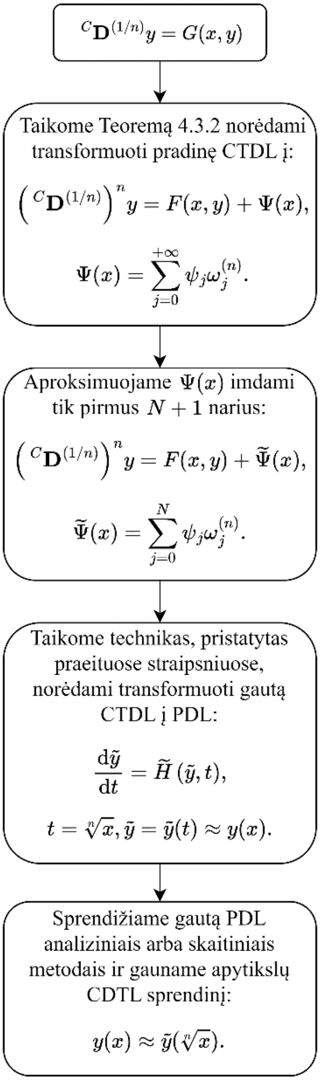
čia

$$\Theta_{y_1}(x) = \sum_{j=0}^{+\infty} \theta_j w_j^{(2)}; \;\; \theta_0 = 0,$$

$$\theta_j = \sum_{k=1}^{j} \frac{1}{\Gamma\left(\frac{k}{2}+1\right)} \left(\frac{\Gamma\left(\frac{j+3}{2}\right)}{\Gamma\left(\frac{j-k+3}{2}\right)} - 2\frac{\Gamma\left(\frac{j}{2}+1\right)}{\Gamma\left(\frac{j-k}{2}+1\right)} \right) \gamma_k \gamma_{j-k+1}, \quad j = 1,2,\ldots.$$

Teoremos pabaiga

Teorema 4.3.2 leidžia CTDL (4.3.39) sprendimui naudoti ankstesniuose straipsniuose išvystytas metodologijas. Tuomet CTDL (4.3.39) sprendinių konstravimas gali būti vykdomas pagal schemą, pateiktą 4.3.3 pav.



4.3.3 pav. ${}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)}$ tipo Caputo trupmeninės diferencialinės lygties sprendimo schema

Išvados

Šiame straipsnyje Caputo trupmeninių laipsninių eilučių sąvoka buvo panaudota ${}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)}$ tipo Caputo trupmeninių diferencialinių lygčių analizei bei sprendimui. Įrodyta, kad, esant tam tikroms sąlygoms, CTDL (4.3.39) gali būti transformuota į $\left({}^{C}\mathbf{D}^{\left(\frac{1}{n}\right)}\right)^{n}$ tipo CTDL (4.3.38), kuri savo ruožtu gali būti sprendžiama taikant anksčiau pristatytas metodologijas. Siūlomo metodo veikimui pademonstruoti pateiktas skaitinis pavyzdys.

4.4. Išvados

Disertacijoje sukurta visiškai nauja Caputo trupmeninių diferencialinių lygčių (CTDL) analizės metodologija. Pristatytos technikos, grindžiamos Caputo trupmeninių laipsninių eilučių koncepcija bei Caputo trupmeninio diferencijavimo operatoriaus algebrine realizacija. Iš esmės pateikta metodika yra operatorinio skaičiavimo apibendrinimas nuo paprastųjų iki trupmeninių diferencialinių lygčių, suteikiantis visiškai naują požiūrį į CTDL skaitinių bei analitinių sprendinių konstravimą. Taikant pasiūlytą metodologiją įvairių tipų CTDL, padarytos šios išvados:

1. Skirtingų eilių Caputo trupmeninių diferencialinių lygčių (CTDL) sprendiniai pasižymi paveldimumu: aukštesnės eilės Caputo trupmeninės lygtys paveldi tam tikrus sprendinius iš žemesnės eilės lygčių, kai dalis pradinių sąlygų yra lygios nuliui.
2. Caputo trupmeninę diferencialinę lygtį su polinominiu netiesiškumu (4.3.13) galima redukuoti į sveikosios eilės paprastąją diferencialinę lygtį (4.3.15), panaudojant Caputo trupmeninių laipsninių eilučių teoriją.
3. CTDL sprendiniai gali būti išplėsti iki neigiamos realiosios pusašės, panaudojant metodologiją, aprašytą 4.3.2 skyriuje. Tokiu atveju gaunami daugiareikšmiai ir kompleksiniai CTDL sprendiniai, atitinkantys neigiamas x reikšmes.
4. Sukurta metodologija, pagrįsta Caputo trupmeninių laipsninių eilučių koncepcija, gali būti panaudota apytiksliams CTDL sprendiniams sudaryti, taikant pasiūlytą naują adaptyvią pusiau analitinę schemą.
5. ${}^{C}\mathbf{D}_x^{\left(\frac{1}{n}\right)}$ tipo CTDL, esant tam tikroms sąlygoms (išvardintoms 4.3.2 teoremoje), gali būti transformuotos į $\left({}^{C}\mathbf{D}^{\left(\frac{1}{n}\right)}\right)^n$ tipo CTDL, kurios savo ruožtu gali būti sprendžiamos taikant disertacijoje pristatytas metodologijas.

REFERENCES

1. NAVICKAS, Z., T. TELKSNYS, R. MARCINKEVICIUS AND M. RAGULSKIS Operator-based approach for the construction of analytical soliton solutions to nonlinear fractional-order differential equations. Chaos, Solitons & Fractals, 2017, 104, 625-634.
2. NAVICKAS, Z., T. TELKSNYS, I. TIMOFEJEVA, R. MARCINKEVIČIUS, et al. An operator-based approach for the construction of closed-form solutions to fractional differential equations. Mathematical Modelling and Analysis, 2018, 23(4), 665-685.
3. MERAL, F., T. ROYSTON AND R. MAGIN Fractional calculus in viscoelasticity: an experimental study. Communications in nonlinear science and numerical simulation, 2010, 15(4), 939-945.
4. SUN, H., Y. ZHANG, D. BALEANU, W. CHEN, et al. A new collection of real world applications of fractional calculus in science and engineering. Communications in nonlinear science and numerical simulation, 2018, 64, 213-231.
5. KUMAR, D. AND D. BALEANU. Fractional calculus and its applications in physics. In.: Frontiers Media SA, 2019, vol. 7, p. 81.
6. HILFER, R. *Applications of fractional calculus in physics*. Edtion ed.: World scientific, 2000. ISBN 9814496200.
7. BALEANU, D., H. M. SRIVASTAVA AND C. CATTANI Recent advances in computational biology. Chaos, Solitons, and Fractals, 2022, 163, 112579.
8. KUMAR, D. AND J. SINGH *Fractional calculus in medical and health science*. Edtion ed.: CRC Press, 2020. ISBN 1000081818.
9. TARASOV, V. E. AND V. V. TARASOVA *Economic dynamics with memory: Fractional calculus approach*. Edtion ed.: Walter de Gruyter GmbH & Co KG, 2021. ISBN 3110624818.
10. BAGLEY, R. L. AND P. TORVIK A theoretical basis for the application of fractional calculus to viscoelasticity. Journal of Rheology, 1983, 27(3), 201-210.
11. LOVOIE, J., T. J. OSLER AND R. TREMBLAY Fractional derivatives and special functions. SIAM review, 1976, 18(2), 240-268.
12. CAPUTO, M. Linear models of dissipation whose Q is almost frequency independent—II. Geophysical Journal International, 1967, 13(5), 529-539.
13. OLDHAM, K. AND J. SPANIER *The fractional calculus theory and applications of differentiation and integration to arbitrary order*. Edtion ed.: Elsevier, 1974. ISBN 0080956203.
14. SINGH, J., A. GUPTA AND D. BALEANU On the analysis of an analytical approach for fractional Caudrey-Dodd-Gibbon equations. Alexandria Engineering Journal, 2022, 61(7), 5073-5082.
15. ALCHIKH, R. AND S. KHURI Numerical solution of a fractional differential equation arising in optics. Optik, 2020, 208, 163911.
16. MING, C., F. LIU, L. ZHENG, I. TURNER, et al. Analytical solutions of multi-term time fractional differential equations and application to unsteady flows of generalized viscoelastic fluid. Computers & Mathematics with Applications, 2016, 72(9), 2084-2097.

17. ASL, M. S. AND M. JAVIDI Novel algorithms to estimate nonlinear FDEs: applied to fractional order nutrient-phytoplankton–zooplankton system. Journal of Computational and Applied Mathematics, 2018, 339, 193-207.
18. FRUNZO, L., R. GARRA, A. GIUSTI AND V. LUONGO Modeling biological systems with an improved fractional Gompertz law. Communications in nonlinear science and numerical simulation, 2019, 74, 260-267.
19. DARASSI, M. H., M. A. KHAN, FATMAWATI AND M. S. ALQARNI Analysis of the Competition System Using Parameterized Fractional Differential Equations: Application to Real Data. Symmetry, 2023, 15(2), 542.
20. LI, X.-P., H. F. ALRIHIELI, E. A. ALGEHYNE, M. A. KHAN, et al. Application of piecewise fractional differential equation to COVID-19 infection dynamics. Results in Physics, 2022, 39, 105685.
21. KARAMAN, M. M., Y. SUI, H. WANG, R. L. MAGIN, et al. Differentiating low-and high-grade pediatric brain tumors using a continuous-time random-walk diffusion model at high b-values. Magnetic resonance in medicine, 2016, 76(4), 1149-1157.
22. MILLER, K. S. AND B. ROSS *An introduction to the fractional calculus and fractional differential equations*, 1993.
23. KILBAS, A. A., H. M. SRIVASTAVA AND J. J. TRUJILLO *Theory and applications of fractional differential equations*. Edtion ed.: elsevier, 2006. ISBN 0444518320.
24. PODLUBNY, I. *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*. Edtion ed.: Elsevier, 1998. ISBN 0080531989.
25. ODIBAT, Z. AND S. MOMANI Application of variational iteration method to nonlinear differential equations of fractional order. International Journal of Nonlinear Sciences and Numerical Simulation, 2006, 7(1), 27-34.
26. DUAN, J.-S., T. CHAOLU, R. RACH AND L. LU The Adomian decomposition method with convergence acceleration techniques for nonlinear fractional differential equations. Computers & Mathematics with Applications, 2013, 66(5), 728-736.
27. VONG, S., P. LYU, X. CHEN AND S.-L. LEI High order finite difference method for time-space fractional differential equations with Caputo and Riemann-Liouville derivatives. Numerical Algorithms, 2016, 72, 195-210.
28. LI, C. AND Z. WANG The local discontinuous Galerkin finite element methods for Caputo-type partial differential equations: Numerical analysis. Applied Numerical Mathematics, 2019, 140, 1-22.
29. ABD-ELHAMEED, W. M. AND Y. H. YOUSSRI A novel operational matrix of Caputo fractional derivatives of Fibonacci polynomials: Spectral solutions of fractional differential equations. Entropy, 2016, 18(10), 345.
30. BROWN, J. W. AND R. V. CHURCHILL *Complex variables and applications*. Edtion ed.: McGraw-Hill, 2009.
31. NAVICKAS, Z. AND L. BIKULČIENE Expressions of solutions of ordinary differential equations by standard functions. Mathematical Modelling and Analysis, 2006, 11(4), 399-412.

32. NAVICKAS, Z., M. RAGULSKIS AND L. BIKULČIENĖ Special solutions of Huxley differential equation. Mathematical Modelling and Analysis, 2011, 16(2), 248-259.
33. NAVICKAS, Z., L. BIKULCIENE, M. RAHULA AND M. RAGULSKIS Algebraic operator method for the construction of solitary solutions to nonlinear differential equations. Communications in nonlinear science and numerical simulation, 2013, 18(6), 1374-1389.
34. GARRAPPA, R. Numerical solution of fractional differential equations: A survey and a software tutorial. Mathematics, 2018, 6(2), 16.

COPIES OF PAPERS INCLUDED IN THIS THESIS

The subsequent pages contain copies of the four papers which form the basis of this thesis.

The papers are as follows:

1. Navickas, Zenonas, et al. The Fractal Structure of Analytical Solutions to Fractional Riccati Equation. *Fractals*, doi: 10.1142/S0218348X23401308
2. Timofejeva, Inga, et al. The Extension of Analytic Solutions to FDEs to the Negative Half-Line. *AIMS Mathematics 6.4* (2021): 3257-3271.
3. Timofejeva, Inga, et al. An Operator-Based Scheme for the Numerical Integration of FDEs. *Mathematics 9.12* (2021): 1372.
4. Marcinkevicius, Romas, et al. The Construction of Solutions to ${}^{C}\mathbf{D}^{(1/n)}$ Type FDEs via Reduction to $\left({}^{C}\mathbf{D}^{\left(\frac{1}{n}\right)}\right)^{n}$ type FDEs. *AIMS Mathematics 7.9* (2022): 16536-16554, doi: 10.3934/math.2022905.

Accepted Manuscript

Fractals

Article Title:	The Fractal Structure of Analytical Solutions to Fractional Riccati Equation
Author(s):	Zenonas Navickas, Tadas Telksnys, Inga Timofejeva, Romas Marcinkevicius, Minvydas Ragulskis
DOI:	10.1142/S0218348X23401308
Received:	28 March 2020
Accepted:	24 February 2022
To be cited as:	Zenonas Navickas *et al.*, The Fractal Structure of Analytical Solutions to Fractional Riccati Equation, *Fractals*, doi: 10.1142/S0218348X23401308
Link to final version:	https://doi.org/10.1142/S0218348X23401308

THE FRACTAL STRUCTURE OF ANALYTICAL SOLUTIONS TO FRACTIONAL RICCATI EQUATION

ZENONAS NAVICKAS[a], TADAS TELKSNYS[a,§], INGA TIMOFEJEVA[a], ROMAS MARCINKEVICIUS[b], MINVYDAS RAGULSKIS[a]

[a] *Center for Nonlinear Systems, Kaunas University of Technology, Studentu 50-147, Kaunas LT-51368, Lithuania*

[b] *Department of Software Engineering, Kaunas University of Technology, Studentu 50-415, Kaunas LT-51368, Lithuania*

Received March 28, 2020
Accepted February 24, 2022

Abstract

Analytical solutions to the fractional Riccati equation are considered in this paper. Solutions to fractional differential equations are expressed in the form of fractional power series in the Caputo algebra. It is demonstrated that solutions to higher-order Riccati fractional equations inherit some solutions from lower-order Riccati equations under special initial conditions. Such nested and fractal-like structure of solutions is investigated by means of analytical fractional differentiation operator techniques and computational experiments.

Keywords: Fractional Differential Equation; Operator Calculus; Analytical Solution

[§]Corresponding Author: Tadas Telksnys; email:tadas.telksnys@ktu.lt

1. INTRODUCTION

Though the concept of fractional-order derivatives dates back to the 17th century, fractional-order calculus has become more prominently used for the modeling of real-world phenomena only in recent years.[1,2] Extensive applications of such models can be encountered in the fields of physics,[3] engineering,[4] biomedicine[5] and image processing.[6] A short review of typical examples concerning the use of fractional calculus in mathematical modeling is given below.

A novel fractional differential and variational model capable of realizing the image fusion, super-resolution, and the edge information enhancement simultaneously has been introduced in.[7] A new framework of nonlocal deformation in non-rigid image registration is developed using fractional Euler-Lagrange equations in.[8] A spatial fractional telegraph equation is used to construct an algorithm for image structure preserving denoising in.[9]

The use of matrix fractional differential equations in economic and quantum physics has been discussed in.[10] It is shown in[11] that fractional-order models are better suited than their integer counterparts in modeling the properties of electrical energy storage devices. A moderate epidemiological model is used for the description of computer viruses with a fractional order derivative having non-singular kernel in.[12] It was demonstrated in[13] that fractional convectiondiffusion equations can capture the gas breakthrough curves including their apparent positive skewness.

A spatial fractional-order thermal transport equation with the Caputo fractional derivative is proposed in[14] to describe convective heat transfer of nanofluids within disordered porous media in boundary layer flow. Viscoelastic constitutive laws for arterial wall mechanics are investigated using fractional order partial differential equations (PDEs) in.[15] A novel variable order fractional differential-based texture enhancement algorithm with applications used in medical imaging is developed in.[16] High-order fractional PDEs are applied to the surface generation of proteins in.[17]

Operator-based approach for the construction of analytic solutions to fractional differential equations is reported in.[18] This technique is based on Caputo algebra of fractional power series and fractional differentiation and integration operators defined on the basis of this algebra. The main objective of this paper is to investigate the fractal structure of such analytic solutions to FDEs.

2. PRELIMINARIES

Main concepts and definitions concerning Caputo fractional power series and operators are presented in this Section. The fractional power series presented here are a generalization of.[18] Note that Caputo fractional differentiation and integration are defined differently than in the classical sense (via integral transformations), but through series basis functions. However, these two approaches yield equivalent results.

2.1. Caputo fractional power series and operators

Let $\frac{1}{n}, n \in \mathbb{N}$ denote the order of the considered fractional derivative. Consider the following sequence of functions:

$$w_j^{(n)} = \frac{x^{\frac{j}{n}}}{\Gamma\left(\frac{j}{n}+1\right)}, \quad j = 0, 1, \ldots. \tag{1}$$

If the derivative order reads $\frac{m}{n}, m < n$, where m, n are coprime natural numbers, then the basis is still defined as in (1).

The following fractional power series are considered in this paper:

$$f = \sum_{j=0}^{+\infty} c_j w_j^{(n)}, \quad c_j \in \mathbb{C}. \tag{2}$$

Series defined by (2) are called Caputo fractional power series. The set of all such series is denoted as ${}^{C}\mathbb{F}_n$. Addition and multiplication of series in this set is performed using conventional operations. Note that $w_l^{(n)} w_k^{(n)} = \binom{\frac{k+l}{n}}{\frac{k}{n}} w_{k+j}^{(n)}$. Given two fractional power series $f = \sum_{j=0}^{+\infty} c_j w_j^{(n)}$ and $g = \sum_{j=0}^{+\infty} b_j w_j^{(n)}$, the product is defined in the Cauchy sense:

$$f \cdot g = \sum_{j=0}^{+\infty} \left(\sum_{r=0}^{j} \binom{\frac{j}{n}}{\frac{r}{n}} c_r b_{j-r} \right) w_j^{(n)}, \tag{3}$$

where

$$\binom{\lambda}{\mu} = \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\lambda-\mu+1)}; \tag{4}$$

denotes the generalized binomial coefficient for $\lambda, \mu \in \mathbb{R}, \lambda \geqslant \mu$.

As proven in,[18] the set ${}^{C}\mathbb{F}_n$ with addition, multiplication and product by a scalar operations forms an algebra ${}^{C}\mathcal{F}_n$ over the field $\mathbb{C}$.

For any function $f \in {}^{C}\mathbb{F}_n$, Caputo integration and differentiation operators are defined by the following equalities:

$$ {}^{C}\mathbf{I}^{(1/n)} w_j^{(n)} = w_{j+1}^{(n)}, \quad j = 0, 1, \ldots; \tag{5} $$

$$ {}^{C}\mathbf{D}^{(1/n)} w_j^{(n)} = \begin{cases} 0, & j = 0; \\ w_{j-1}^{(n)}, & j = 1, 2, \ldots \end{cases} \tag{6} $$

Fractional derivatives and integrals of order $\frac{m}{n}$ are represented by powers of the respective operators: $\left({}^{C}\mathbf{I}^{(1/n)}\right)^m, \left({}^{C}\mathbf{D}^{(1/n)}\right)^m$.

3. SOLUTION OF THE FRACTIONAL RICCATI EQUATION

As mentioned in the Introduction, the main objective of this paper is to explore the fractal structure of analytic solutions expressible in the form of Caputo fractional power series. Without loss of generality, the following fractional Riccati equation is considered:

$$ \left({}^{C}\mathbf{D}^{(1/n)}\right)^n y_n = a_2 y_n^2 + a_1 y_n + a_0, \tag{7} $$

where $a_k \in \mathbb{C}$ and $y_n = y_n\left(x; s_0^{(n)}, s_1^{(n)}, \ldots, s_{n-1}^{(n)}\right) \in {}^{C}\mathcal{F}_n$. The parameters $s_0^{(n)}, \ldots, s_{n-1}^{(n)}$ correspond to initial conditions formulated at $x = 0$:

$$ \left.\left({}^{C}\mathbf{D}^{(1/n)}\right)^k y_n\right|_{x=0} = s_k^{(n)}; \quad k = 0, \ldots, n-1. \tag{8} $$

As noted in the previous section, if non-integers powers of the series y_n are considered, the operator $\left({}^{C}\mathbf{D}^{(1/n)}\right)^n$ is not identical to $\frac{\mathrm{d}}{\mathrm{d}x}$, because the former operator is applied to the Caputo power series comprised of basis elements $w_0^{(n)}, w_1^{(n)}, \ldots$, while the latter acts on power series containing only integer powers of x.

In the remainder of this section, the solution to (7) is derived by computing first the series coefficients of the solution y_n. Once the recursive relation that defines the series coefficients is known, a generating function for these coefficients can be defined via an ordinary differential equation. The solution to this equation is a transformation of the solution to (7).

3.1. Derivation of recurrence relations for the coefficients of the solution to the fractional Riccati equation

Since $y_n \in {}^{C}\mathcal{F}_n$, it has the power series form:

$$ y_n = \sum_{j=0}^{+\infty} c_j w_j^{(n)}; \quad c_j \in \mathbb{C}. \tag{9} $$

The definition of operators ${}^{C}\mathbf{D}^{(1/n)}$ and conventional algebraic operations with power series yield the following identities:

$$ \left({}^{C}\mathbf{D}^{(1/n)}\right)^n y_n = \sum_{j=0}^{+\infty} c_{j+n} w_j^{(n)}; \tag{10} $$

$$ y_n^2 = \sum_{j=0}^{+\infty}\left(\sum_{r=0}^{j}\binom{\frac{j}{n}}{\frac{r}{n}} c_r c_{j-r}\right) w_j^{(n)} = \sum_{j=0}^{+\infty}\left(\sum_{r=0}^{j}\frac{\Gamma\left(\frac{j}{n}+1\right)}{\Gamma\left(\frac{r}{n}+1\right)\Gamma\left(\frac{j-r}{n}+1\right)} c_r c_{j-r}\right) w_j^{(n)}. \tag{11} $$

Inserting (9)–(11) into (7) and collecting like terms results in:

$$ \sum_{j=0}^{+\infty} c_{j+n} w_j^{(n)} = \sum_{j=0}^{+\infty}\left(a_2\left(\sum_{r=0}^{j}\frac{c_r}{\Gamma\left(\frac{r}{n}+1\right)}\frac{c_{j-r}}{\Gamma\left(\frac{j-r}{n}+1\right)}\right) \cdot \Gamma\left(\frac{j}{n}+1\right) + a_1 c_j + \delta_j a_0\right) w_j^{(n)}, \tag{12} $$

where $\delta_j = 1$ if $j = 0$ and zero otherwise.

Equation (12) yields a recursive relation for the coefficients of the solution y_n:

$$ c_{j+n} = a_2\left(\sum_{r=0}^{j}\frac{c_r}{\Gamma\left(\frac{r}{n}+1\right)}\frac{c_{j-r}}{\Gamma\left(\frac{j-r}{n}+1\right)}\right)\Gamma\left(\frac{j}{n}+1\right) + a_1 c_j + \delta_j a_0, \tag{13} $$

for $j = 0, 1, \ldots$. To simplify (13), the following transformation is used:

$$ \gamma_j^{(n)} = \frac{c_j}{\Gamma\left(\frac{j}{n}+1\right)}; \quad j = 0, 1, \ldots. \tag{14} $$

Rearranging (13) results in:

$$(j+n)\,\gamma_{j+n}^{(n)} = n\left(a_2\sum_{r=0}^{j}\left(\gamma_r^{(n)}\gamma_{j-r}^{(n)}\right) + a_1\gamma_j^{(n)} + \right.$$
$$\left. +\delta_j a_0\right); \quad j=0,1,\ldots \tag{15}$$

Note that the first n coefficients $\gamma_0^{(n)},\ldots,\gamma_{n-1}^{(n)}$ can be chosen arbitrarily. Thus the series y_n is made to conform to initial conditions (8). Since the definition of operator ${}^C\mathbf{D}^{(1/n)}$ yields that $c_k = s_k^{(n)}; k = 0,1,\ldots,n-1$, the following relation holds:

$$s_k^{(n)} = \Gamma\left(\frac{k}{n}+1\right)\gamma_k^{(n)}, \quad k=0,1,\ldots,n-1. \tag{16}$$

Thus, for a given set of initial conditions, the first n coefficients of recurrence sequence $\gamma_j^{(n)}$ is computed using $s_0^{(n)},\ldots,s_{n-1}^{(n)}$ and further iterated via the formula (15).

3.2. Characteristic function of the sequence $(\gamma_j^{(n)}; j = 0,1,\ldots)$ and its generating equation

The characteristic function of sequence $(\gamma_j^{(n)}; j = 0,1,\ldots)$ reads:

$$\varphi_n(t) = \sum_{j=0}^{+\infty}\gamma_j^{(n)}t^j. \tag{17}$$

Multiplying both sides of (15) by t^j and summing from $j=0$ results in the equality:

$$\sum_{j=0}^{+\infty}(j+n)\,\gamma_{j+n}^{(n)}t^j = n\left(a_2\sum_{j=0}^{+\infty}\left(\sum_{r=0}^{j}\left(\gamma_r^{(n)}\gamma_{j-r}^{(n)}\right)t^j\right) + \right.$$
$$\left. +a_1\sum_{j=0}^{+\infty}\gamma_j^{(n)}t^j + a_0\right). \tag{18}$$

Applying (17) to (18) yields:

$$\sum_{j=n}^{+\infty} j\gamma_j^{(n)}t^{j-n} = n\left(a_2\varphi_n^2(t) + a_1\varphi_n(t) + a_0\right). \tag{19}$$

Note that the left hand side of (19) can be rewritten as:

$$\sum_{j=n}^{+\infty} j\gamma_j^{(n)}t^{j-n} = \frac{1}{t^{n-1}}\left(\sum_{j=1}^{+\infty} j\gamma_j^{(n)}t^{j-1} - \sum_{j=1}^{n-1} j\gamma_j^{(n)}t^{j-1}\right)$$
$$= \frac{1}{t^{n-1}}\left(\frac{\mathrm{d}\varphi_n}{\mathrm{d}t} - \sum_{j=1}^{n-1} j\gamma_j^{(n)}t^{j-1}\right). \tag{20}$$

Inserting (20) into (19) and simplifying yields an ordinary differential equation with respect to the generating function φ_n:

$$\frac{\mathrm{d}\varphi_n}{\mathrm{d}t} = nt^{n-1}\left(a_2\varphi_n^2(t) + a_1\varphi_n(t) + a_0\right) +$$
$$+ \sum_{j=1}^{n-1} j\gamma_j^{(n)}t^{j-1}, \quad n = 2,3,\ldots. \tag{21}$$

Note that for the non-fractional Riccati equation $(n=1)$, the equation (21) and the Riccati equation itself coincide.

3.3. Solution of the fractional Riccati equation via generating function φ_n

The function $\varphi_n(t)$ can be utilized to express solutions to (7). First, note that $c_j = \Gamma\left(\frac{j}{n}+1\right)\gamma_j^{(n)}, j = 0,1,\ldots$. Then y_n can be written as:

$$y_n = \sum_{j=0}^{+\infty} c_j w_j^{(n)} = \sum_{j=0}^{+\infty}\Gamma\left(\frac{j}{n}+1\right)\gamma_j^{(n)}\frac{x^j n}{\Gamma\left(\frac{j}{n}+1\right)} =$$
$$= \sum_{j=0}^{+\infty}\gamma_j^{(n)}x^{\frac{j}{n}} = \varphi_n\left(\sqrt[n]{x}\right). \tag{22}$$

Also, note that:

$$\frac{\mathrm{d}y_n}{\mathrm{d}x} = \left.\frac{\mathrm{d}\varphi_n}{\mathrm{d}t}\right|_{t=\sqrt[n]{x}} \cdot \frac{1}{n}x^{\frac{1-n}{n}}, \tag{23}$$

which leads to:

$$\left.\frac{\mathrm{d}\varphi_n}{\mathrm{d}t}\right|_{t=\sqrt[n]{x}} = nx^{\frac{n-1}{n}}\frac{\mathrm{d}y_n}{\mathrm{d}x}. \tag{24}$$

Evaluating both sides of (21) at $t = \sqrt[n]{x}$ yields:

$$nx^{\frac{n-1}{n}}\frac{\mathrm{d}y_n}{\mathrm{d}x} = nx^{\frac{n-1}{n}}\left(a_2y_n^2 + a_1y_n + a_0\right) + \sum_{j=1}^{n-1} j\gamma_j^{(n)}x^{\frac{j-1}{n}}, \tag{25}$$

that can be further simplified into:

$$n x^{\frac{n-1}{n}} \left(\frac{\mathrm{d}y_n}{\mathrm{d}x} - a_2 y_n^2 - a_1 y_n - a_0 \right) = \sum_{j=1}^{n-1} j \gamma_j^{(n)} x^{\frac{j-1}{n}}. \tag{26}$$

Equation (26) yields the following result:

Remark

Let $\psi_n(x) := \sum_{j=0}^{n-1} \gamma_j^{(n)} x^{\frac{j}{n}}$. Then, the initial value problem on the Riccati fractional differential equation (7), (8) is equivalent to the initial value problem on the following ordinary differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}x} (y_n - \psi_n) = a_2 y_n^2 + a_1 y_n + a_0; \tag{27}$$

$$y_n(0) = s_0^{(n)} = \gamma_0^{(n)}. \tag{28}$$

Corollary

Since the expressions for y_n and ψ_n are known, (27), (28) can be rewritten as:

$$\sum_{j=n}^{+\infty} \frac{j}{n} \gamma_j^{(n)} x^{\frac{j-n}{n}} = \sum_{j=0}^{+\infty} \left(a_2 \sum_{r=0}^{j} \gamma_r^{(n)} \gamma_{j-r}^{(n)} + a_1 \gamma_j^{(n)} \right) + a_0. \tag{29}$$

Note that (29) is equivalent to (15).

It can be observed from (26) that some solutions to the fractional Riccati equation remain viable for any value of n. If the initial conditions are set to $s_1^{(n)} = s_2^{(n)} = \cdots = s_{n-1}^{(n)} = 0$, then the right hand side of (26) vanishes and, futhermore $\gamma_j^{(n)} \neq 0$ only if $j = kn$ for some $k \in \mathbb{N}$. In that case, the solution $y_n \in {}^{C}\mathcal{F}_1$ and satisfies the ordinary Riccati equation:

$$\frac{\mathrm{d}y_n}{\mathrm{d}x} = a_2 y_n^2 + a_1 y_n + a_0. \tag{30}$$

This observation leads to the conclusion that for any $n \in \mathbb{N}$, the fractional Riccati equation (7) inherits the non-fractional solutions of (30) for some initial conditions. Furthermore, to showcase the fractal nature of fractional differential equations, this argument can be extended: any equation (7) with order $n = km, \quad k, m \in \mathbb{N}$ inherits solution from the fractional equation with orders $n = k$ and $n = m$.

4. COMPUTATIONAL EXPERIMENTS

In this section, the fractal nature of the analyzed fractional differential equations will be demonstrated using numerical experiments. Let us consider the following fractional Riccati equation:

$$\left({}^{C}\mathbf{D}^{(1/n)} \right)^n y_n = y_n^2 + y_n - 6. \tag{31}$$

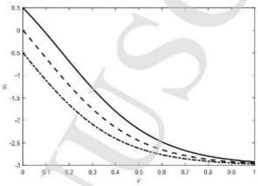


Fig. 1 Kink solutions to the non-fractional Riccati equation (31) for $n = 1$. The solid, dashed and dash-dotted lines correspond to initial conditions $s_0^{(1)} = \frac{1}{2}, 0$ and $-\frac{1}{2}$ respectively.

A comparison of numerical integration results for the fractional Riccati differential equation (31) of different orders ($n = 1, 2, 3$) is presented below.

The non-fractional Riccati equation (31) with $n = 1$ admits only well-known kink solutions,[19,20] depicted in Fig. 1. Note that the number of initial conditions on (31) increases as n grows, thus yielding a larger set of solutions. A comparison of solutions to the non-fractional and fractional Riccati equations (for $n = 1$ and $n = 2$ respectively) is shown in Fig. 2.

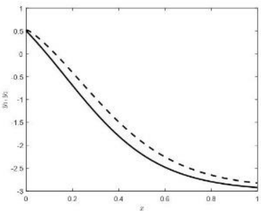


Fig. 2 Solutions to equation (31) for $n = 1$ (solid line) and $n = 2$ (dashed line). Initial conditions are set to $s_0^{(1)} = s_0^{(2)} = s_1^{(2)} = \frac{1}{2}$.

To compare numerical solutions to (31) with different values of n, the following difference measure is introduced:

$$\Delta_{n,m}\left(s_1^{(n)},\ldots,s_{n-1}^{(n)};s_1^{(m)},\ldots,s_{m-1}^{(m)}\right)= \\ =\sum_{j=0}^{N}\left(\widehat{y}_n\left(jh;s_1^{(n)},\ldots,s_{n-1}^{(n)}\right)- \right. \\ \left. -\widehat{y}_m\left(jh;s_1^{(m)},\ldots,s_{m-1}^{(m)}\right)\right)^2, \tag{32}$$

where $\widehat{y}_k\left(x;s_1^{(k)},\ldots,s_{k-1}^{(k)}\right)$ is the numerical solution to (31) of order k with initial conditions $s_1^{(k)},\ldots,s_{k-1}^{(k)}$; h is the constant integrator step-size (the classical Runge-Kutta 4th order method was used in the computations); N is the number of time-forward steps. Initial conditions $s_0^{(n)}$ and $s_0^{(m)}$ are set to be equal.

The plot of $\Delta_{1,2}\left(s_1^{(2)}\right)$ when $s_0^{(1)}=s_0^{(2)}=\frac{1}{2}$ is shown in Fig. 3. It can be seen that y_2 coincides with the kink solution of the non-fractional Riccati equation when $s_1^{(2)}=0$, which verifies the analytical results presented in the previous Section.

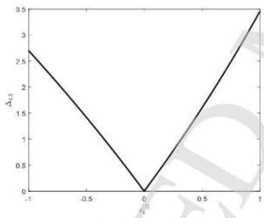


Fig. 3 Plot of $\Delta_{1,2}\left(s_1^{(2)}\right)$ for $s_0^{(1)}=s_0^{(2)}=\frac{1}{2}$. Note that the fractional and non-fractional solutions coincide when $s_1^{(2)}=0$.

An analogous experiment was performed to compare solutions of the non-fractional Riccati equation and fractional equation of order $n=3$. The initial conditions were set to $s_0^{(1)}=s_0^{(3)}=\frac{1}{2}$. The contour plot for $\Delta_{1,3}\left(s_1^{(3)},s_2^{(3)}\right)$ is given in Fig. 4. It can be observed that the kink solution of the non-fractional equation satisfies the fractional equation of order $n=3$, but only if initial conditions $s_1^{(3)}=s_2^{(3)}$ are equal to zero. This result is further clarified in Fig. 5, where the section of the contour plot along the line $s_1^{(3)}=0$ is given. However, it must be noted that solutions to the non-fractional and fractional equations only coincide for a single pair of initial conditions $s_1^{(3)}=s_2^{(3)}=0$. A plot of solutions y_1,y_3 for initial conditions not on the minimum point $(0,0)$ is given in Fig. 6.

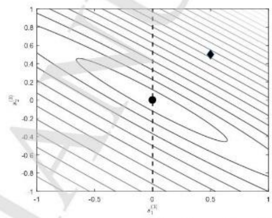


Fig. 4 Contour plot of $\Delta_{1,3}\left(s_1^{(3)},s_2^{(3)}\right)$ for $s_0^{(1)}=s_0^{(3)}=\frac{1}{2}$. Solid lines indicate contour lines of $\Delta_{1,3}$; the black circle denotes the minimum point $s_1^{(3)}=s_2^{(3)}=0$ where $\Delta_{1,3}=0$; the dashed line corresponds to a plot of $\Delta_{1,3}\left(0,s_2^{(3)}\right)$ shown in 5; the diamond corresponds to the initial conditions used to plot comparison of solutions y_1 and y_3 given in 6.

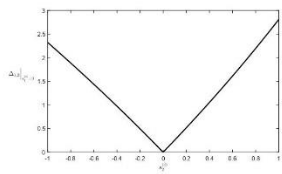


Fig. 5 Plot of $\Delta_{1,3}\left(0,s_2^{(3)}\right)$. Note that the solutions coincide on $s_2^{(3)}=0$.

From the presented results it follows that solutions of the fractional Riccati equations of orders $n=2$ and $n=3$ coincide for initial conditions

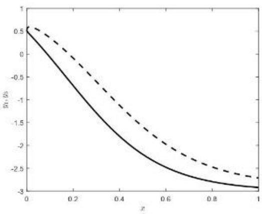


Fig. 6 Plot of solutions y_1 (solid line) and y_3 (dashed line) for initial conditions $s_0^{(1)} = s_0^{(3)} = \frac{1}{2}$, $s_1^{(3)} = s_2^{(3)} = \frac{1}{2}$.

$s_1^{(2)} = s_1^{(3)} = s_2^{(3)} = 0$. These results also hold true for higher order of fractional differential equations, which means that the non-fractional kink solution satisfies any fractional equation. Furthermore, a similar argument can be used to show that a fractional Riccati differential equation 7 of order n admits all solutions (when some initial conditions are set to zero) of the same equation with order m if m divides n.

5. FRACTAL STRUCTURE OF ANALYTICAL SOLUTIONS TO FRACTIONAL RICCATI EQUATION

Computational experiments presented in the previous Section indicate that the solutions to the fractional Riccati equation exhibit a nested, fractal-like structure in which solutions of lower-order equations satisfy higher-order equations if some initial conditions are set to zero.

Let p, q be relatively prime natural numbers. Consider Caputo power series algebras ${}^C\mathcal{F}_{pm}, {}^C\mathcal{F}_{qm}$, where $m \in \mathbb{N}$. The definition of fractional power series yields the following properties:

$$ {}^C\mathcal{F}_{pm} \cap {}^C\mathcal{F}_{qm} = {}^C\mathcal{F}_m; \tag{33} $$

and

$$ {}^C\mathcal{F}_{pm} \cup {}^C\mathcal{F}_{qm} \subseteq {}^C\mathcal{F}_{pqm}. \tag{34} $$

The relationship between different orders of fractional power series is illustrated with more detail in Fig. 7. It is clear that the basis elements corresponding to different orders n of fractional differential equations may intersect. Thus, as demonstrated via numerical experiments in the previous Section, solutions from a higher-order equation q may inherit solutions from a lower-order equation $p < q$ under some initial conditions.

Consider two fractional Riccati equations (7), (8) of orders p and q. Let $g := \gcd(p, q)$ and define $s^{(p)} := \frac{p}{g}, s^{(q)} := \frac{q}{g}$. Note that $s^{(p)}, s^{(q)} \in \mathbb{N}$ and the basis of order p, q intersect at powers of $x^{\frac{k}{g}}, k = 0, 1, \ldots$ if the following conditions on coefficients $\gamma_j^{(p)}, \gamma_j^{(q)}$ hold true:

$$ \gamma_j^{(p)} = 0, \quad j \neq s^{(p)} l; \tag{35} $$

$$ \gamma_i^{(q)} = 0, \quad i \neq s^{(q)} l, \tag{36} $$

where $l = 0, 1, \ldots$.

6. CONCLUSIONS

The fractal structure of solutions to fractional differential equations has been investigated in this paper. It is shown that the fractional Riccati equation can be solved by considering the recurrence relations between the coefficients of fractional power series. It is proven that the generating function of the series coefficients satisfies an associated ordinary differential equation and can be used to construct the solution to the fractional Riccati equation.

Furthermore, it appears that solutions to fractional equations exhibit a nested, fractal-like structure, which is investigated via computational experiments and theoretical investigation of the fractional power series basis. This fractality property results in the fact that higher-order fractional Riccati equations inherit some solutions from lower-order equations when a subset of initial conditions is set to zero.

Further investigation of the fractal properties and construction of analytical solutions of fractional differential equations remains a definite objective of future research.

REFERENCES

1. X.-J. Yang, F. Gao, and J. Yang, *General fractional derivatives with applications in viscoelasticity* (Academic Press, Cambridge, 2020).
2. X.-J. Yang, *General fractional derivatives: theory, methods and applications* (CRC Press, Boca Raton, 2019).

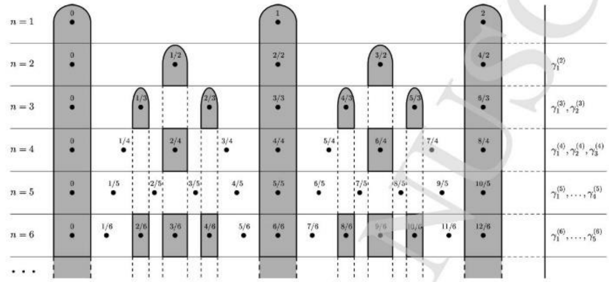


Fig. 7 Fractal-like structure of fractional power series basis. Each row $n = k; k = 1, 2, \ldots$ displays basis elements $w_j^{(k)}; j = 0, 1, \ldots$ of the fractional power series of order k. Accordingly, fractions $\frac{j}{k}$ correspond to powers of x for each base element $w_j^{(k)}; j = 0, 1, \ldots$, respectively. Parameters $\gamma_v^{(k)}; v = 1, \ldots, k-1$ on the right represent the set of arbitrarily chosen coefficients of the corresponding ODE (21). Gray-filled columns (or sections of columns) correspond to the same power of x in respective base elements.

3. H. Richard, *Fractional Calculus: An Introduction for Physicists* (World Scientific, Singapore, 2014).
4. D. Xue, *Fractional-Order Control Systems. Fundamentals and Numerical Implementations* (De Gruyter, Berlin, 2017).
5. R. L. Magin, *Fractional Calculus in Bioengineering* (Begell House Redding, Danburry, 2006).
6. Q. Yang, D. Chen, T. Zhao, and Y. Chen, Fractional calculus in image processing: a review, *Fractional Calculus and Applied Analysis* **19** (2016), 1222–1249.
7. H. Li, Z. Yu, and C. Mao, Fractional differential and variational method for image fusion and super-resolution, *Neurocomputing* **171** (2016), 138–148.
8. J. Zhang and K. Chen, Variational image registration by a total fractional-order variation model, *Journal of Computational Physics* **293** (2015), 442–461.
9. W. Zhang, J. Li, and Y. Yang, Spatial fractional telegraph equation for image structure preserving denoising, *Signal Processing* **107** (2015), 368–377.
10. V. E. Tarasov and V. V. Tarasova, Time-dependent fractional dynamics with memory in quantum and economic physics, *Annals of Physics* **383** (2017), 579–599.
11. M. E. Fouda, A. S. Elwakil, A. G. Radwan, and A. Allagui, Power and energy analysis of fractional-order electrical energy storage devices, *Energy* **111** (2016), 785–792.
12. J. Singh, D. Kumar, Z. Hammouch, and A. Atangana, A fractional epidemiological model for computer viruses pertaining to a new fractional derivative, *Applied Mathematics and Computation* **316** (2018), 504–515.
13. A. Chang, H. Sun, C. Zheng, B. Lu, C. Lu, R. Ma, and Y. Zhang, A time fractional convection–diffusion equation to model gas transport through heterogeneous soil and gas reservoirs *Physica A: Statistical Mechanics and its Applications* **502** (2018), 356–369.
14. M. Pan, L. Zheng, F. Liu, C. Liu, and X. Chen, A spatial-fractional thermal transport model for nanofluid in porous media *Applied Mathematical Modelling* **53** (2018), 622–634.
15. Y. Yu, P. Perdikaris, and G. E. Karniadakis, Fractional modeling of viscoelasticity in 3D cerebral arteries and aneurysms *Journal of Computational Physics* **323** (2016), 219–242.
16. Q. Yu, V. Vegh, F. Liu, and I. Turner, A variable order fractional differential-based texture enhancement algorithm with application in medical imaging *PloS one* **10** (2015), e0132952.
17. L. Hu, D. Chen, and G.-W. Wei, High-order fractional partial differential equation transform for molecular surface construction *Molecular based mathematical biology* **1** (2013), 1–25.
18. Z. Navickas, T. Telksnys, R. Marcinkevicius, and M. Ragulskis, Operator-based approach for the con-

struction of analytical soliton solutions to nonlinear fractional-order differential equations *Chaos Solitons and Fractals* **104** (2017), 625–634.
19. N. A. Kudryashov, Simplest equation method to look for exact solutions of nonlinear differential equations *Chaos Soliton Fractals* **24** (2005), 1217–1231.
20. Z. Navickas, L. Bikulciene, and M. Ragulskis, Generalization of Exp-function and other standard function methods *Applied Mathematics and Computation* **216** (2010), 2380–2393.

AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(4): 3257–3271.
DOI:10.3934/math.2021195
Received: 14 October 2020
Accepted: 31 December 2020
Published: 15 January 2021

Research article

The extension of analytic solutions to FDEs to the negative half-line

Inga Timofejeva[1], Zenonas Navickas[1], Tadas Telksnys[1,*], Romas Marcinkevičius[2], Xiao-Jun Yang[3] and Minvydas Ragulskis[1]

[1] Center for Nonlinear Systems, Kaunas University of Technology, Studentu 50-147, Kaunas LT-51368, Lithuania

[2] Department of Software Engineering, Kaunas University of Technology, Studentu 50-415, Kaunas LT-51368, Lithuania

[3] School of Mathematics and State Key Laboratory for Geomechanics and Deep Underground Engineering, China University of Mining and Technology, No.1, Daxue Road, Xuzhou 221116, Jiangsu, P. R. China

* **Correspondence:** Email: tadas.telksnys@ktu.lt; Tel: +370 636 44870.

Abstract: An analytical framework for the extension of solutions to fractional differential equations (FDEs) to the negative half-line is presented in this paper. The proposed technique is based on the construction of a special characteristic equation corresponding to the original FDE (when the characteristic equation does exist). This characteristic equation enables the construction analytic solutions to FDEs are expressed in the form of infinite fractional power series. Necessary and sufficient conditions for the existence of such an extension are discussed in detail. It is demonstrated that the extension of solutions to FDEs to the negative half-line is not a single-valued operation. Computational experiments are used to illustrate the efficacy of the proposed scheme.

Keywords: fractional differential equation; operator calculus; negative half-line; solitary wave; inverse balancing
Mathematics Subject Classification: 34A08, 34A25, 26A33

1. Introduction

Even though the roots of fractional calculus are in the late 17th century [1], the subject has received significant attention from researchers only in recent decades. The importance of fractional calculus and fractional differential equations (FDEs) is evidenced by the emergence of fractional-order models for real-world phenomena, which has lead to a plethora of applications of FDEs [2]. A short review of the areas in which FDEs are applied is given below.

The advantage of fractional-order models is that they can be used to model systems with the memory effect, under which the current state of the system at time $t = t_0$ is affected by an interval of states at time $t < t_0$. Thus, such models lend themselves naturally to the description of viscoelastic materials [3, 4]. It can be noted that most biological tissues possess viscoelastic properties [5] which has lead to the use of FDEs to describe the evolution of various biomedical systems [6, 7, 8]. In physics, fractional-order models are widely used to study dielectric materials [9, 10, 11] and Bose-Einstein condensates [12]. FDEs have also been used to describe the memory effect in economic models [13, 14].

Due to the applicability of fractional-order models, a wide variety of methods have been developed to construct solutions to FDEs. The Q-homotopy analysis transform is applied in order to obtain analytical solutions to the fractional coupled Ramani equation in [15]. The solutions of integer-order differential equations are used to construct the solutions to the models of cooling and spread of epidemic diseases [16]. An Adams-type predictor-corrector method for the construction of numerical solutions to FDEs is discussed in [17]. A new family of predictor-corrector methods for FDEs is discussed in [18]. A Legendre operational matrix approach is used to numerically solve FDEs in [19, 20].

The homotopy analysis method is adapted to FDEs for the computation of approximate solutions to various equations, including fractional Korteweg-de-Vries, Burgers and Boussinesq models in [21]. A class of rational Krylov methods for the construction of numerical solutions to partial fractional differential equations is considered in [22]. The traditional reproducing kernel method is adapted to be applicable to FDEs in [23]. A restricted transform technique is used to solve irrational order FDEs in [24]. Artificial neural networks are used to construct approximate solutions to FDEs in [25].

The main objective of this paper is to present a novel framework for the extension of the solutions to fractional differential equation (FDEs) to negative argument values. The main idea of this procedure is to construct the ordinary differential equation (ODE) that has a solution which can be transformed to the solution of the considered FDE. Since the solution of the ODE also exists for negative argument values, it can be used to extend the solution of the FDE for such argument values as well. This results in complex-valued solutions to FDEs for negative argument values. Such extension is a novel viewpoint into the solutions to FDEs, as they can be analyzed not only in the set $x \in \mathbb{R}_+$, but in the entire real line $x \in \mathbb{R}$. This goes beyond the current state-of-the-art, as with current algorithms the solutions to FDEs cannot be considered in the negative x-axis.

2. Preliminaries

2.1. *Fractional power series and functions*

In this paper we consider functions defined via power series:

$$f(x) = \sum_{j=0}^{+\infty} v_j w_j^{(n)}, \tag{2.1}$$

where $v_j \in \mathbb{R}$ are coefficients of the series and the base functions $w_j^{(n)}; n \in \mathbb{N}, j = 0, 1, \ldots$ are defined as:

$$w_j^{(n)}(x) = \frac{x^{\frac{j}{n}}}{\Gamma\left(\frac{j}{n} + 1\right)}; n \in \mathbb{N}, j = 0, 1, \ldots. \tag{2.2}$$

The usual notation $\Gamma(x)$ is used to denote the gamma function [26].

The number n which denotes the order of the basis of fractional power series is selected with respect to the fractional derivative order which is analyzed. If the fractional derivative is of order $\alpha = \frac{k}{m}$, and $\gcd(k, m) = 1$, it is taken that $n = m$ [27].

Consider $x > 0$ and let $\sqrt[n]{x} \in \mathbb{R}$. Using the substitution $t := \sqrt[n]{x}$, (2.1) can be rewritten as:

$$f(x) = \widehat{f}(t) = \sum_{j=0}^{+\infty} \gamma_j \left(\sqrt[n]{x}\right)^j = \sum_{j=0}^{+\infty} \gamma_j t^j, \quad \gamma_j = \frac{v_j}{\Gamma\left(\frac{j}{n} + 1\right)}. \tag{2.3}$$

For subsequent computations it is required that $\widehat{f}(t)$ be analytic (except for some finite number of singularity points) and non-singular at $t = 0$. Thus, there exists a convergence radius $|t| < T_0$; $T_0 > 0$. The function $\widehat{f}$ will be referred to as the characteristic function of f.

For any given $n \in \mathbb{N}$, the set of all functions defined by (2.1) and meeting the requirements stated above is denoted as ${}^C\mathbb{F}_n$. Conventional addition and multiplication operations on functions belonging to ${}^C\mathbb{F}_n$ are defined. For example, given two series $f = \sum_{j=0}^{+\infty} v_j w_j^{(n)}$ and $g = \sum_{j=0}^{+\infty} b_j w_j^{(n)}$, the product is defined as:

$$f \cdot g = \sum_{j=0}^{+\infty} \left(\sum_{k=0}^{j} \binom{j}{k} v_k b_{j-k} \right) w_j^{(n)}. \tag{2.4}$$

Note that the sum above obeys the finite summation principle: all coefficients of the above series are given by finite sums, not series [28].

The set ${}^C\mathbb{F}_n$ with addition and multiplication operations forms an algebra over $\mathbb{C}$. The properties of algebra ${}^C\mathbb{F}_n$ are discussed in detail in [27] and [29].

2.2. *Caputo fractional differentiation operator*

Let $f = \sum_{j=0}^{+\infty} v_j w_j^{(n)} \in {}^C\mathbb{F}_n$. Caputo differentiation operator of order $\frac{1}{n}$ is then defined as [30]:

$${}^C\mathbf{D}^{(1/n)} f = \sum_{j=0}^{+\infty} v_{j+1} w_j^{(n)}. \tag{2.5}$$

Note the Caputo derivatives of basis functions [27]:

$${}^C\mathbf{D}^{(1/n)} w_j^{(n)} = \begin{cases} 0, & j = 0 \\ w_{j-1}^{(n)}, & j = 1, 2, \ldots \end{cases} \tag{2.6}$$

Derivatives of rational order $\frac{k}{n}$; $k \in \mathbb{N}$ are defined by higher powers of operator ${}^C\mathbf{D}^{(1/n)}$ [29]:

$$\left({}^C\mathbf{D}^{(1/n)}\right)^k f = \sum_{j=0}^{+\infty} v_{j+k} w_j^{(n)} = \sum_{j=0}^{+\infty} \frac{\Gamma\left(\frac{j+k}{n} + 1\right) \gamma_{j+k}}{\Gamma\left(\frac{j}{n} + 1\right)} \left(\sqrt[n]{x}\right)^j. \tag{2.7}$$

In the special case of $k = m \cdot n$; $m \in \mathbb{N}$, (2.7) reads:

$$\begin{aligned} \left({}^C\mathbf{D}^{(1/n)}\right)^{m \cdot n} f &= \sum_{j=0}^{+\infty} \frac{\Gamma\left(\frac{j}{n} + m + 1\right)}{\Gamma\left(\frac{j}{n} + 1\right)} \gamma_{j+mn} \left(\sqrt[n]{x}\right)^j \\ &= \sum_{j=0}^{+\infty} \left(\prod_{k=1}^{m} \left(\frac{j}{n} + k\right) \right) \gamma_{j+mn} \left(\sqrt[n]{x}\right)^j. \end{aligned} \tag{2.8}$$

2.3. *Riemann extension algorithm for Caputo functions*

It is well-known that analytic functions can be extended beyond their convergence radius [31]. The same idea is adapted for series of the type (2.1). Let $f = \sum_{j=0}^{+\infty} \gamma_j \left(\sqrt[n]{x}\right)^j$ and choose $\sqrt[n]{x_0}$ such that x_0 is nonzero and in the characteristic function's convergence radius, $0 < x_0 < T_0^n$. Then (2.1) can be rearranged:

$$\begin{aligned} f &= \sum_{j=0}^{+\infty} \gamma_j \left(\left(\sqrt[n]{x} - \sqrt[n]{x_0}\right) + \sqrt[n]{x_0}\right)^j = \sum_{j=0}^{+\infty} \sum_{k=0}^{j} \binom{j}{k} \gamma_j \left(\sqrt[n]{x_0}\right)^{j-k} \left(\sqrt[n]{x} - \sqrt[n]{x_0}\right)^k \\ &= \sum_{k=0}^{+\infty} \left(\sum_{j=k}^{+\infty} \binom{j}{k} \gamma_j \left(\sqrt[n]{x_0}\right)^{j-k} \right) \left(\sqrt[n]{x} - \sqrt[n]{x_0}\right)^k = \sum_{j=0}^{+\infty} \widehat{\gamma}_j \left(\sqrt[n]{x} - \sqrt[n]{x_0}\right)^j . \end{aligned} \tag{2.9}$$

Coefficients $\widehat{\gamma}_j$ are defined as:

$$\widehat{\gamma}_j = \sum_{k=j}^{+\infty} \binom{k}{j} \gamma_k \left(\sqrt[n]{x_0}\right)^{k-j} . \tag{2.10}$$

Note that coefficients (2.10) are finite, because the non-extended series f converges in its convergence radius $0 < x_0 < T_0^n$. The convergence radius for (2.9) is $|\sqrt[n]{x} - \sqrt[n]{x_0}| < T_1$, $T_1 > 0$. Furthermore, if the same procedure is repeated for $x_1 \neq x_0$ that also satisfies $0 < x_1 < T_0^n$, then:

$$\sum_{j=0}^{+\infty} \widehat{\gamma}_j \left(\sqrt[n]{x} - \sqrt[n]{x_0}\right)^j = \sum_{j=0}^{+\infty} \widetilde{\gamma}_j \left(\sqrt[n]{x} - \sqrt[n]{x_1}\right)^j , \tag{2.11}$$

as long as x is such that series on both sides of the equation converge.

This procedure can be applied to extend a function from the set ${}^C\mathbb{F}_n$ between two singularity points (or singularity and infinity, if no other singularities are present) by rewriting the function in different basis $\left(\sqrt[n]{x} - \sqrt[n]{x_k}\right)^j$, $k = 0, 1, \ldots$. Note that these extended series coincide for x that lies in all of their convergence radii, thus the extensions are unique.

Note that computational operations, such as the Caputo differentiation operator are only defined for basis $\left(\sqrt[n]{x}\right)^j$ (in the neighbourhood of $x = 0$). For this reason, all computations are first performed in this neighbourhood and then extended throughout the entire function domain. Thus, the zero neighbourhood of x is called the algebraic operation neighbourhood. The concept of this neighbourhood is introduced in [28].

2.4. *Generalized differential operator technique*

In the subsequent sections of this paper, constructing series solutions to ordinary differential equations (ODEs) is necessary. The generalized differential operator technique is used for this task. A short description of this method is given in this section. A detailed overview can be found in [32, 33].

Let us consider an ordinary differential equation in the explicit form:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = P(t, z); \quad z(c) = s; \quad c, s \in \mathbb{R}, \tag{2.12}$$

where P is an arbitrary bivariate analytic function. The following generalized differential operator is associated with the ODE (2.12) [34]:

$$\mathbf{D}_{cs} = \mathbf{D}_c + P(c, s)\mathbf{D}_s, \tag{2.13}$$

where $\mathbf{D}_\lambda$ denotes the partial differentiation operator with respect to variable λ.

The series solution to (2.12) can then be written in the form [32]:

$$z(t, c, s) = \sum_{j=0}^{+\infty} \frac{(t-c)^j}{j!} p_j(c, s), \tag{2.14}$$

where $p_j(c, s) = \mathbf{D}_{cs}^j s$.

3. Solutions to fractional differential equations with polynomial nonlinearity

The following fractional differential equation is considered in this section:

$$\left({}^C\mathbf{D}^{(1/n)}\right)^n y_n = Q_m(y_n); \quad y_n = y_n(x), \tag{3.1}$$

where $y_n \in {}^C\mathbb{F}_n$ and Q_m is an arbitrary mth order polynomial:

$$Q_m(y) = \sum_{k=0}^{m} a_k y^k; \quad m \in \mathbb{N},\ a_m \neq 0. \tag{3.2}$$

In the remainder of this section, solutions in the operation neighbourhood of $x = 0$ will be constructed directly and later extended into different neighbourhoods using the procedure described in previous sections.

3.1. Construction of solutions to (3.1) in the neighbourhood of $x = 0$

Consider a series $y_n = \sum_{j=0}^{+\infty} v_j w_j^{(n)} = \sum_{j=0}^{+\infty} \gamma_j \left(\sqrt[n]{x}\right)^j \in {}^C\mathbb{F}_n$. Inserting the function into (3.1) yields:

$$\sum_{j=0}^{+\infty}\left(\frac{j}{n}+1\right)\gamma_{j+n}\left(\sqrt[n]{x}\right)^j = \sum_{k=1}^{m}\left(a_k\left(\sum_{j=0}^{+\infty}\left(\sum_{k_1+\cdots+k_m=j}\prod_{l=1}^{k}\gamma_{k_l}\right)\left(\sqrt[n]{x}\right)^j\right)\right) + a_0. \tag{3.3}$$

Let $\sqrt[n]{x} = t$ and $\widehat{y}_n = \sum_{j=1}^{+\infty} \gamma_j t^j$. Then, (3.3) yields:

$$\sum_{j=0}^{+\infty}(j+n)\gamma_{j+n}t^j = nQ_m\left(\widehat{y}_n\right). \tag{3.4}$$

Rearranging the sums in (3.4) yields:

$$\sum_{j=n}^{+\infty} j\gamma_j t^{j-1} = nt^{n-1}Q_m\left(\widehat{y}_n\right). \tag{3.5}$$

It can be noted that adding $\sum_{j=1}^{n-1} j\gamma_j t^{j-1}$ to both sides of (3.5) transforms the right side to an ordinary derivative of $\widehat{y}_n$:

$$\frac{\mathrm{d}\widehat{y}_n}{\mathrm{d}t} = nt^{n-1}Q_m + \sum_{j=1}^{n-1} j\gamma_j t^{j-1}. \tag{3.6}$$

Noting that $\gamma_j = \frac{v_j}{\Gamma\left(\frac{j}{n}+1\right)}$ yields a different form of (3.6):

$$\frac{\mathrm{d}\widehat{y}_n}{\mathrm{d}t} = n\left(t^{n-1}Q_m\left(\widehat{y}_n\right) + \sum_{j=1}^{n-1}\frac{v_j}{\Gamma\left(\frac{j}{n}+1\right)}t^{j-1}\right). \tag{3.7}$$

Adding the initial condition $y_n(c) = s$, Equation (3.7) can be solved using the method described in 2.4. The generalized differential operator for (3.7) reads:

$$\mathbf{D}_{cs} = \mathbf{D}_c + n\left(c^{n-1}Q_m(s) + \sum_{j=1}^{n-1}\frac{v_j}{\Gamma\left(\frac{j}{n}+1\right)}c^{j-1}\right)\mathbf{D}_s. \tag{3.8}$$

The solution to (3.7) (taking values of t for which the series converges) has the form:

$$\widehat{y}_n = \widehat{y}_n(t, c, s) = \sum_{j=0}^{+\infty}\frac{(t-c)^j}{j!}p_j(c, s), \tag{3.9}$$

where $p_j(c, s) = \mathbf{D}_{cs}^j s$. Noting that $y_n(x) = \widehat{y}_n\left(\sqrt[n]{x}\right)$ yields Theorem 3.1.

Theorem 3.1. *The fractional differential equation (3.1) admits the following general solution for any parameter values* $u_0; v_1, v_2, \ldots, v_{n-1}; x_0 \geq 0$*:*

$$y_n(x) = \sum_{j=0}^{+\infty}\frac{\left(\sqrt[n]{x} - \sqrt[n]{x_0}\right)^j}{j!}p_j\left(\sqrt[n]{x_0}, u_0\right), \tag{3.10}$$

where

$$y_n\left(\sqrt[n]{x_0}\right) = u_0, \tag{3.11}$$

and

$$\left({}^C\mathbf{D}^{(1/n)}\right)^k y_n\Big|_{x=0} = v_k; \quad k = 1, \ldots, n-1, \tag{3.12}$$

if x satisfies $\left|\sqrt[n]{x} - \sqrt[n]{x_0}\right| < T_{x_0}$. Here $T_{x_0} > 0$ denotes the convergence radius of (3.10).

Note that the results of this section can be extended to analytic functions. Taking $Q(x) = \sum_{k=0}^{+\infty} a_k x^k$ instead of $Q_m(x)$ and following the steps outlined above would yield a theorem that is analogous to Theorem 3.1. However, the analysis of such equations is not the focus of this paper, thus we will consider only polynomial $Q_m(x)$ in the remainder of the text.

3.2. *Cauchy initial value problems on (3.1) formulated at the origin point*

Based on Theorem 3.1, the Cauchy initial value problem for (3.1) can be formulated at point $x = 0$:

$$\left({}^{C}\mathbf{D}^{(1/n)}\right)^{n} y_n = Q_m\left(y_n\right), \quad y_n = y_n\left(x, u_0^{(0)}\right); \tag{3.13}$$

$$\left({}^{C}\mathbf{D}^{(1/n)}\right)^{k} y_n\Big|_{x=0} = v_k; \quad k = 1, \ldots, n-1; \tag{3.14}$$

$$y_n\Big|_{x=0} = u_0^{(0)}. \tag{3.15}$$

Note that since $\sqrt[n]{x}$ in general possesses n branches, in these computations we select the branch with a minimum value of $\arg x$.

By the results obtained in the previous section, in the neighbourhood of $x = 0$, the problem (3.13)–(3.15) has the solution:

$$y_n = \sum_{j=0}^{+\infty} \frac{\left(\sqrt[n]{x}\right)^j}{j!} p_j\left(0, u_0^{(0)}\right). \tag{3.16}$$

This solution can be extended by applying the algorithm discussed in section 2.3. First, choose a sequence $x_0 = 0 < x_1 < x_2 < \ldots$ and freely choose $u_0^{(0)} \in \mathbb{R}$. Then, compute parameters $u_0^{(1)}, u_0^{(2)}, \ldots$ as:

$$u_0^{(k+1)} = \widehat{y}_n\left(\sqrt[n]{x_{k+1}}, \sqrt[n]{x_k}, u_0^{(k)}\right), \quad k = 0, 1, \ldots \tag{3.17}$$

The general solution to (3.13)–(3.15) can then be written for any k:

$$y_n = y_n\left(x, u_0^{(0)}\right) = \widehat{y}_n\left(\sqrt[n]{x}, \sqrt[n]{x_k}, u_0^{(k)}\right), \tag{3.18}$$

for $\left|\sqrt[n]{x} - \sqrt[n]{x_k}\right| < T_{x_k}$, $T_{x_k} > 0, x_k \geq 0$.

3.3. *Cauchy initial value problems on (3.1) not formulated at the origin point*

The rearrangements described in the previous section can be applied to construct a more general Cauchy initial value problem, where the first initial condition can be formulated at nonzero x:

$$\left({}^{C}\mathbf{D}^{(1/n)}\right)^{n} y_n = Q_m\left(y_n\right), \quad y_n = y_n\left(x, u_0^{(0)}\right); \tag{3.19}$$

$$\left({}^{C}\mathbf{D}^{(1/n)}\right)^{k} y_n\Big|_{x=0} = v_k; \quad k = 1, \ldots, n-1; \tag{3.20}$$

$$y_n\Big|_{x=x_0} = u_0^{(0)}; \quad x_0 \neq 0.. \tag{3.21}$$

Analogously to problem (3.13)–(3.15), the solution reads:

$$y_n = y_n\left(x, x_0, u_0^{(0)}\right) = \widehat{y}_n\left(\sqrt[n]{x}, \sqrt[n]{x_0}, u_0^{(0)}\right). \tag{3.22}$$

Note that initial conditions that correspond to fractional derivative values still must be formulated at $x = 0$ for solution (3.22) to hold true.

4. Solutions to fractional differential equations with polynomial nonlinearity for negative values of x

As stated earlier, the main objective of this paper is to extend the solutions to FDEs so that they would exist for negative values of argument x. This extension results in complex-valued fractional power series, which are defined in the following subsection. Note that this extension is not possible when considering fractional derivatives in the traditional Caputo sense, as they are defined via integrals only for non-negative values of x.

4.1. Complex fractional power series

Let us consider an extension to the power series basis presented in section 2.1:

$$\left(w_j^{(n)}\right)_k := \frac{\left(\sqrt[n]{|x|}\right)^j}{\Gamma\left(\frac{j}{n}+1\right)} \exp\left(ij\frac{\arg x + 2\pi k}{n}\right), \tag{4.1}$$

where $x \in \mathbb{R}$, $k = 0, \ldots, n-1$; $j = 0, 1, \ldots$ and $\sqrt[n]{|x|} \in \mathbb{R}$ denotes the root with the lowest value of $\arg x$. The basis with $k = 0$ corresponds to $w_j^{(n)}$ presented in section 2.1, while $k = 1, 2, \ldots, n-1$ result in complex-valued functions $\left(w_j^{(n)}\right)_k$.

Using (4.1), the standard fractional power series (2.1) can be extended into n complex-valued series:

$$f_k(x) = \sum_{j=0}^{+\infty} v_j \left(w_j^{(n)}\right)_k; \quad k = 0, \ldots, n-1. \tag{4.2}$$

4.2. Extension of solutions to FDEs with polynomial nonlinearity

By Theorem 3.1, the solution to

$$\left({}^C\mathbf{D}^{(1/n)}\right)^n y_n = Q_m(y_n); \quad y_n = y_n(x), \tag{4.3}$$

reads:

$$y_n(x) = \sum_{j=0}^{+\infty} \frac{\left(\sqrt[n]{x} - \sqrt[n]{x_0}\right)^j}{j!} p_j\left(\sqrt[n]{x_0}, u_0\right), \tag{4.4}$$

where coefficients p_j are generated via the generalized differential operator technique as described in previous sections. Using the basis defined in the previous section, the solution (4.4) can extended into the complex plane:

$$(y_n(x))_k = \sum_{j=0}^{+\infty} \frac{\left(\sqrt[n]{|x|} - \sqrt[n]{|x_0|}\right)^j}{j!} \exp\left(ij\frac{\arg x_0 + 2\pi k}{n}\right) p_j\left(\sqrt[n]{|x_0|} \exp\left(i\frac{\arg x_0 + 2\pi k}{n}\right), u_0\right). \tag{4.5}$$

Denoting $\alpha := \arg x_0$ and $\beta_n^{(k)}(\alpha) := \exp\left(i\frac{\alpha+2\pi k}{n}\right)$, series (4.5) can be rewritten as:

$$(y_n(x))_k = \sum_{j=0}^{+\infty} \frac{\left(\sqrt[n]{|x|} - \sqrt[n]{|x_0|}\right)^j}{j!} \left(\lambda_j^{(k)}(|x_0|, \alpha, u_0) + i\mu_j^{(k)}(|x_0|, \alpha, u_0)\right), \tag{4.6}$$

where

$$\lambda_j^{(k)}(|x_0|, \alpha, u_0) := \mathrm{Re}\left(\left(\beta_n^{(k)}(\alpha)\right)^j p_j\left(\sqrt[n]{|x_0|}\beta_n^{(k)}(\alpha), u_0\right)\right); \tag{4.7}$$

$$\mu_j^{(k)}(|x_0|, \alpha, u_0) := \mathrm{Im}\left(\left(\beta_n^{(k)}(\alpha)\right)^j p_j\left(\sqrt[n]{|x_0|}\beta_n^{(k)}(\alpha), u_0\right)\right). \tag{4.8}$$

Expression (4.6) allows the consideration of solutions for $x < 0$, however, these solutions are multi-valued (a total of n branches corresponding to the number of distinct roots $\sqrt[n]{x}$) and take complex values.

4.3. Riemann extension scheme for solutions of FDEs with negative argument values

A refinement of the procedure described in section 3.2 is needed to construct particular solutions to FDEs (4.3). Two sequences of numbers are chosen: $\cdots < x_{-r} < \cdots < x_{-1} < 0$; $0 < x_1 < x_2 < \ldots$ and $x_0 = 0$. For any $k = 0, \ldots, n-1$, the initial condition $s_0^{(k)} \in \mathbb{R}$ can be taken freely.

Sequences $s_{-r}^{(k)}$ and $s_r^{(k)}$ are computed using relations:

$$s_{r+1}^{(k)} := \widehat{y}\left(\left(\sqrt[n]{x_{r+1}}\right)_k, \left(\sqrt[n]{x_r}\right)_k, s_r^{(k)}\right); \quad r = 0, 1, \ldots; \tag{4.9}$$

$$s_{-r-1}^{(k)} := \widehat{y}\left(\left(\sqrt[n]{x_{-r-1}}\right)_k, \left(\sqrt[n]{x_{-r}}\right)_k, s_{-r}^{(k)}\right); \quad r = 0, 1, \ldots. \tag{4.10}$$

Here $\widehat{y}$ denotes the solution of the ODE that the FDE (4.3) is transformed to, as given by Theorem 3.1.

Then the particular solution to (4.3) corresponding to the kth branch of the root $\sqrt[n]{x}$ reads:

$$(y_n(x))_k = \widehat{y}\left(\left(\sqrt[n]{x}\right)_k; \left(\sqrt[n]{x_r}\right)_k; s_r^{(k)}\right), \tag{4.11}$$

where $r = 0, \pm 1, \pm 2, \ldots$.

Note that the sequence $\cdots < x_{-1} < x_0 < x_1 < \ldots$ must be chosen in such a way that the series generated by $\widehat{y}$ are convergent in both $\mathbb{R}$ and $\mathbb{C}$. Also, the point $x = 0$ is a branching point for the solution to the FDE, thus the functions $(y_n(x))_k$ are non-differentiable at this point.

5. Computational experiments

Let us consider the non-fractional Riccati equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2y^2 - 5y - 3; \tag{5.1}$$

$$y(x_0) = u_0. \tag{5.2}$$

It is well-known that (5.1) admits the kink solitary solution:

$$y(x, x_0, u_0) = \frac{\alpha_2(u_0 - \alpha_1)\exp(2\alpha_1(x - x_0)) - \alpha_1(u_0 - \alpha_2)\exp(2\alpha_2(x - x_0))}{(u_0 - \alpha_1)\exp(2\alpha_1(x - x_0)) - (u_0 - \alpha_2)\exp(2\alpha_2(x - x_0))}, \tag{5.3}$$

where $\alpha_1 = 3, \alpha_2 = -\frac{1}{2}$ are the roots of polynomial $2y^2 - 5y - 3$. Solution (5.3) is depicted in Figure 1.

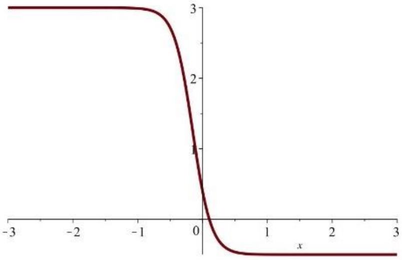


Figure 1. Kink solitary solution to (5.1) for initial conditions $x_0 = 0, u_0 = 2/5$.

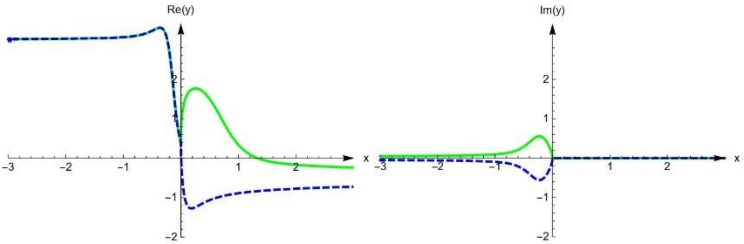
Suppose that $n = 2$, $x_0 = 0$ and let us consider the fractional Riccati equation:

$$\left({}^{C}\mathbf{D}^{(1/2)}\right)^2 y = 2y^2 - 5y - 3. \tag{5.4}$$

As stated earlier, the initial conditions read:

$${}^{C}\mathbf{D}^{(1/2)}y\Big|_{x=0} = v_1; \quad y(0) = u_0^{(0)}. \tag{5.5}$$

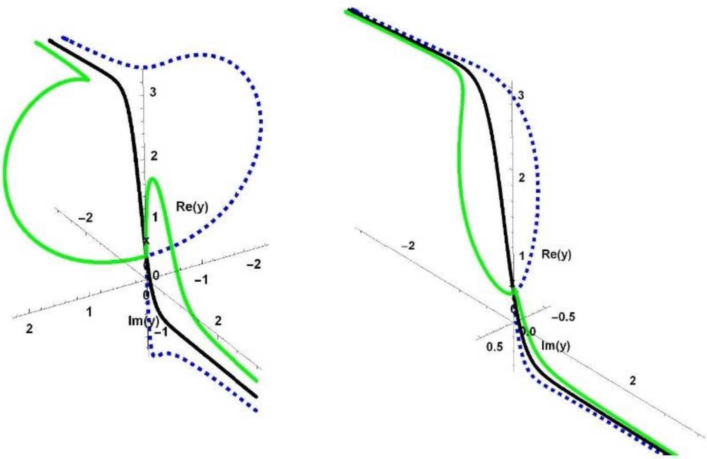
The solution to (5.4), (5.5) is constructed using the algorithm described in section 4.3. Note that since $\sqrt{x}$ possesses two branches, two solutions exist. The real and imaginary parts of these solutions are shown in Figure 2. Note that solution is purely real for $x \geq 0$ – the imaginary parts appears only for negative values of x.



(a) (b)

Figure 2. Real (a) and imaginary (b) parts of the solution to the fractional Riccati equation (5.4) for initial conditions $x_0 = 0, u_0^{(0)} = 2/5, v_1 = 5$. The green solid line and blue dashed line denote solutions corresponding to different branches of $\sqrt{x}$.

It can be seen in Figure 3 that as v_1 approaches zero, the solutions to (5.4) approach the kink solitary solution (5.3).



(a) (b)

Figure 3. Solutions to the fractional Riccati equation (5.4). Initial conditions are set to $x_0 = 0, u_0^{(0)} = 2/5$ for both (a) and (b). The initial condition corresponding to the coefficients of the fractional series powers is set to $v_1 = 5, v_1 = 1$ in (a) and (b) respectively. The green solid line and blue dashed line denote solutions corresponding to different branches of $\sqrt{x}$. The black line corresponds to the non-fractional solitary solution (5.3).

Taking $n = 3$ results in the fractional Riccati equation:

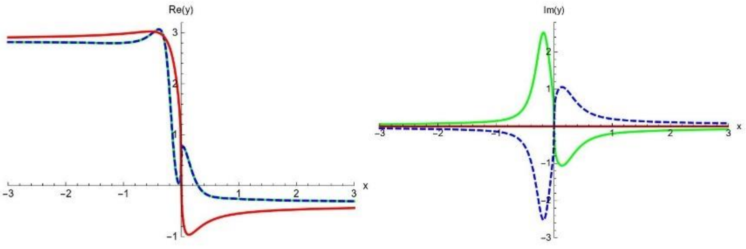
$$\left({}^{C}\mathbf{D}^{(1/3)} \right)^3 y = 2y^2 - 5y - 3. \tag{5.6}$$

The number of initial conditions for (5.6) increases to three:

$${}^{C}\mathbf{D}^{(1/3)} y \Big|_{x=0} = v_1; \quad \left({}^{C}\mathbf{D}^{(1/3)} \right)^2 y \Big|_{x=0} = v_2; \quad y(0) = u_0^{(0)}. \tag{5.7}$$

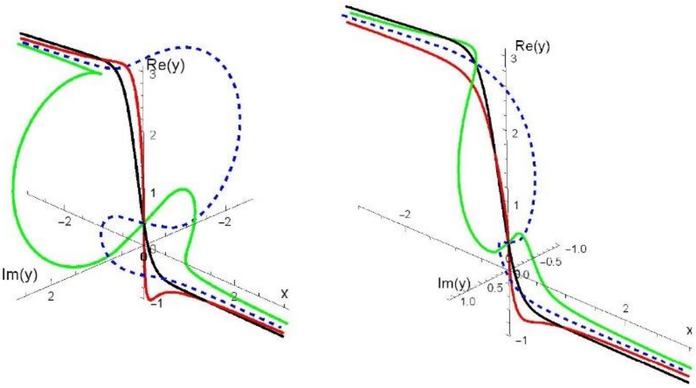
Riccati equation (5.6) has three solutions: two of them are complex-valued and one is a real-valued solution. The real and imaginary parts of these solutions are depicted in Figure 4. A spatial plot for two different sets of initial conditions is shown in Figure 5.

Note that the as $x \to \pm\infty$, the fractional solitary solutions approach the non-fractional kink solitary solution. This is due to the properties of the characteristic equation (3.6). Thus, the limits of the fractional and non-fractional coincide as $x \to \pm\infty$.



(a) (b)

Figure 4. Real (a) and imaginary (b) parts of the solution to the fractional Riccati equation (5.6) for initial conditions $x_0 = 0, u_0^{(0)} = 2/5, v_1 = -1, v_2 = \frac{1}{10}$. The green and red solid lines and blue dashed line denote solutions corresponding to different branches of $\sqrt[3]{x}$. Note that the solution corresponding to the red solid line is real-valued.



(a) (b)

Figure 5. Solutions to the fractional Riccati equation (5.6). Initial conditions are set to $x_0 = 0, u_0^{(0)} = 2/5$ for both (a) and (b). The initial condition corresponding to the coefficients of the fractional series powers is set to $v_1 = -1, v_2 = \frac{1}{10}$ in (a) and $v_1 = -\frac{1}{2}, v_2 = \frac{1}{2}$ in (b). The green and red solid lines and blue dashed line denote solutions corresponding to different branches of $\sqrt[3]{x}$. The black line corresponds to the non-fractional solitary solution (5.3). Note that the solution corresponding to the red solid line is real-valued, but does not coincide with the the non-fractional solitary solution.

6. Conclusions

An analytical framework for the extension of solutions to FDEs into the negative half-line is presented in this paper. The main idea of this extension is the construction of a characteristic differential equation for a given FDE. The solution to the characteristic equation is used to construct a fractional power series solution to the FDE. This solution is then extended using classical Riemann extension techniques to enable the consideration of the solution at a neighbourhood different than the origin $x = 0$. Furthermore, this extension is valid for both positive and negative values of x, which extends the solution to the entire real line.

As demonstrated by computational experiments, the solution to the FDE is complex-valued for negative values of x. Furthermore, there exist n branches of solutions, where n is the denominator of the fractional differentiation order. If the initial conditions (3.20) that correspond to fractional derivatives tend to zero, the solution to the FDE tends to the non-fractional solution of the same equation.

The presented technique provides a solid foundation for the construction of solutions to FDEs on the negative half-line. In particular, this allows to travel backwards in time from the initial conditions of a FDE – such a possibility has not been reported previously. It is well-known that fractional order derivatives help to model memory effects in dynamical systems. Therefore it would be tempting to extend the available coronavirus pandemic models [35, 36] by introducing fractional derivatives. However, the results of this paper show that such an extension would eliminate the possibility of the unique backwards extrapolation. These questions as well as the physical interpretation of the extensions to the negative half-line remain a definite objective of future research.

Acknowledgments

This research was funded by a grant (No. S-COV-20-8) from the Research Council of Lithuania.

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. B. Ross, The development of fractional calculus 1695–1900, *Hist. Math.*, **4** (1977), 75–89.
2. A. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Vol. 204, Elsevier Science Limited, 2006.
3. R. L. Bagley, P. Torvik, A theoretical basis for the application of fractional calculus to viscoelasticity, *J. Rheol.*, **27** (1983), 201–210.
4. K. Adolfsson, M. Enelund, P. Olsson, On the fractional order model of viscoelasticity, *Mech. Time-depend. Mat.*, **9** (2005), 15–34.
5. S. Pal, Mechanical properties of biological materials, *Design of Artificial Human Joints & Organs*, (2014), 23–40. Springer.

6. N. Grahovac, M. Žigić, Modelling of the hamstring muscle group by use of fractional derivatives, *Comput. Math. Appl.*, **59** (2010), 1695–1700.

7. C. Ionescu, A. Lopes, D. Copot, J. T. Machado, J. Bates, The role of fractional calculus in modeling biological phenomena: A review, *Commun. Nonlinear Sci.*, **51** (2017), 141–159.

8. M. P. Aghababa, M. Borjkhani, Chaotic fractional-order model for muscular blood vessel and its control via fractional control scheme, *Complexity*, **20** (2014), 37–46.

9. A. Ortega, J. Rosales, J. Cruz-Duarte, M. Guia, Fractional model of the dielectric dispersion, *Optik*, **180** (2019), 754–759.

10. V. E. Tarasov, Fractional integro-differential equations for electromagnetic waves in dielectric media, *Theor. Math. Phys.*, **158** (2009), 355–359.

11. B. Ducharne, B. Newell, G. Sebald, A unique fractional derivative operator to simulate all dynamic piezoceramic dielectric manifestations: From aging to frequency-dependent hysteresis, *IEEE T. Ultrason. Ferr.*, **67** (2019), 197–206.

12. R. Zhang, Z. Han, Y. Shao, Z. Wang, Y. Wang, The numerical study for the ground and excited states of fractional Bose–Einstein condensates, *Comput. Math. Appl.*, **78** (2019), 1548–1561.

13. V. E. Tarasov, V. V. Tarasova, Macroeconomic models with long dynamic memory: Fractional calculus approach, *Appl. Math. Comput.*, **338** (2018), 466–486.

14. V. E. Tarasov, V. V. Tarasova, Time-dependent fractional dynamics with memory in quantum and economic physics, *Ann. Phys.*, **383** (2017), 579–599.

15. A. A. Arafa, A. M. S. Hagag, A new analytic solution of fractional coupled Ramani equation, *Chinese J. Phys.*, **60** (2019), 388–406.

16. E. Demirci, N. Ozalp, A method for solving differential equations of fractional order, *J. Comput. Appl. Math.*, **236** (2012), 2754–2762.

17. K. Diethelm, N. J. Ford, A. D. Freed, A predictor-corrector approach for the numerical solution of fractional differential equations, *Nonlinear Dynam.*, **29** (2002), 3–22.

18. M. Kumar, V. Daftardar-Gejji, A new family of predictor-corrector methods for solving fractional differential equations, *Appl. Math. Comput.*, **363** (2019), 124633.

19. A. Saadatmandi, M. Dehghan, A new operational matrix for solving fractional-order differential equations, *Comput. Math. Appl.*, **59** (2010), 1326–1336.

20. M. Pourbabaee, A. Saadatmandi, A novel Legendre operational matrix for distributed order fractional differential equations, *Appl. Math. Comput.*, **361** (2019), 215–231.

21. M. Dehghan, J. Manafian, A. Saadatmandi, Solving nonlinear fractional partial differential equations using the homotopy analysis method, *Numerical Methods for Partial Differential Equations: An International Journal*, **26** (2010), 448–479.

22. L. Aceto, D. Bertaccini, F. Durastante, P. Novati, Rational Krylov methods for functions of matrices with applications to fractional partial differential equations, *J. Comput. Phys.*, **396** (2019), 470–482.

23. Y.-L. Wang, L.-n. Jia, H.-l. Zhang, Numerical solution for a class of space-time fractional equation by the piecewise reproducing kernel method, *Int. J. Comput. Math.*, **96** (2019), 2100–2111.

24. A. R. Khudair, S. Haddad, Restricted fractional differential transform for solving irrational order fractional differential equations, *Chaos, Solitons & Fractals*, **101** (2017), 81–85.

25. C. Zúñiga-Aguilar, H. Romero-Ugalde, J. Gómez-Aguilar, R. Escobar-Jiménez, M. Valtierra-Rodriguez, Solving fractional differential equations of variable-order involving operators with Mittag-Leffler kernel using artificial neural networks, *Chaos, Solitons & Fractals*, **103** (2017), 382–403.

26. G. E. Andrews, R. Askey, R. Roy, *Special functions*, Cambridge University Press, Cambridge, 2001.

27. Z. Navickas, T. Telksnys, R. Marcinkevicius, M. Ragulskis, Operator-based approach for the construction of analytical soliton solutions to nonlinear fractional-order differential equations, *Chaos Solitons and Fractals*, **104** (2017), 625–634.

28. L. Bikulčienė, Z. Navickas, Expressions of solutions of linear partial differential equations using algebraic operators and algebraic convolution, *International Conference on Numerical Analysis and Its Applications*, (2008), 208–215. Springer.

29. Z. Navickas, T. Telksnys, I. Timofejeva, R. Marcinkevicius, M. Ragulskis, An operator-based approach for the construction of closed-form solutions to fractional differential equations, *Math. Model. Anal.*, **23** (2018), 665–685.

30. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Netherlands, 2006.

31. J. W. Brown, R. V. Churchill, *Complex variables and applications eighth edition*, Boston: McGraw-Hill Higher Education, 2009.

32. Z. Navickas, L. Bikulciene, Expressions of solutions of ordinary differential equations by standard functions, *Math. Model. Anal.*, **11** (2006), 399–412.

33. Z. Navickas, M. Ragulskis, L. Bikulciene, Special solutions of Huxley differential equation, *Math. Model. Anal.*, **16** (2011), 248–259.

34. Z. Navickas, L. Bikulciene, M. Rahula, M. Ragulskis, Algebraic operator method for the construction of solitary solutions to nonlinear differential equations, *Commun. Nonlinear Sci.*, **18** (2013), 1374–1389.

35. F. Ndairou, I. Area, J. J. Nieto, D. F. Torres, Mathematical modeling of COVID-19 transmission dynamics with a case study of Wuhan, *Chaos, Solitons & Fractals*, (2020), 109846.

36. G. Giordano, F. Blanchini, R. Bruno, P. Colaneri, A. Di Filippo, A. Di Matteo, et al., Modelling the COVID-19 epidemic and implementation of population-wide interventions in Italy, *Nat. Med.*, (2020), 1–6.

mathematics

Article

An Operator-Based Scheme for the Numerical Integration of FDEs

Inga Timofejeva [1], Zenonas Navickas [1], Tadas Telksnys [1], Romas Marcinkevicius [2] and Minvydas Ragulskis [1,*]

[1] Center for Nonlinear Systems, Kaunas University of Technology, Studentu, 50-147 Kaunas, Lithuania; inga.timofejeva@ktu.lt (I.T.); zenonas.navickas@ktu.lt (Z.N.); tadas.telksnys@ktu.lt (T.T.)
[2] Department of Software Engineering, Kaunas University of Technology, Studentu, 50-415 Kaunas, Lithuania; romas.marcinkevicius@ktu.lt
* Correspondence: minvydas.ragulskis@ktu.lt; Tel.: +370-698-22456

Abstract: An operator-based scheme for the numerical integration of fractional differential equations is presented in this paper. The generalized differential operator is used to construct the analytic solution to the corresponding characteristic ordinary differential equation in the form of an infinite power series. The approximate numerical solution is constructed by truncating the power series, and by changing the point of the expansion. The developed adaptive integration step selection strategy is based on the controlled error of approximation induced by the truncation. Computational experiments are used to demonstrate the efficacy of the proposed scheme.

Keywords: fractional differential equation; numerical integration; generalized differential operator

Citation: Timofejeva, I.; Navickas, Z.; Telksnys, T.; Marcinkevicius, R.; Ragulskis, M. An Operator-Based Scheme for the Numerical Integration of FDEs. *Mathematics* **2021**, *9*, 1372. https://doi.org/10.3390/math9121372

Academic Editors: Charampos Tsitouras and Theodore E. Simos

Received: 24 May 2021
Accepted: 10 June 2021
Published: 13 June 2021

1. Introduction

Fractional differential equations (FDEs) play an important role in many research fields. From classical applications of FDEs in modeling viscoelasticity [1,2], to engineering problems [3,4], to more novel fields for the subject such as medical research [5,6] and economics [7,8], FDEs are becoming increasingly widespread. It is natural that a wider usage of fractional-order models has led to a growing interest in numerical integration of FDEs. Some examples of recent research are given below.

A numerical integration technique based on converting the FDE into a set consisting of integral and algebraic equations is presented in [9]. A recursive algorithm based on the Laplace decomposition is used to construct semi-analytical solutions to a Ray-tracing equation in [10]. A new scheme for the construction of numerical solutions that can be applied to several types of fractional derivatives is discussed in [11]. The Ritz approximation is applied to construct numerical solutions to the fractional Fokker–Planck equation in [12]. An approach based on Chebyshev polynomials with time-dependent coefficients is employed to construct numerical solutions to Caputo-type time–space fractional partial differential equations with variable coefficients in [13].

Müntz polynomials are used in conjunction with the collocation to develop a scheme for the numerical integration of FDEs in [14]. Chebyshev polynomials are used in a similar scheme in [15]. The infinite state representation of the Caputo derivative is used in [16] to develop an algorithm for the numerical integration of FDEs. A number of approaches applying wavelets to obtain numerical solutions to FDEs have been considered in [17,18]. A survey of current methods and a collection of software for the integration of FDEs, including explicit, implicit and predictor–corrector methods can be found in [19].

The main objective of this paper is to present a novel FDE integration scheme based on the generalized differential operator technique. The presented technique consists of constructing piecewise-polynomial approximations to the solutions of FDEs via power series. Using these approximations, integration with an adaptive step-size is performed to construct the numerical solution.

Riccati-type equations have recently been discussed in a number of publications concerned with presenting novel FDE integration schemes. He's variational method is applied to the fractional Riccati equation in [20]. A novel homotopy perturbation technique is applied to fractional Riccati models in [21]. A modification of the homotopy perturbation method is used in [22] to construct numerical solutions to the Riccati-type FDEs.

As an example, the following form of the fractional Riccati equation is considered in developing the numerical FDE integration strategy:

$$x\left({}^{C}\mathbf{D}^{1/2}\right)^{2}y = a_0 + a_1 y + a_2 y^2; \quad a_0, a_1, a_2 \in \mathbb{R}, \tag{1}$$

where ${}^{C}\mathbf{D}^{1/2}$ denotes the Caputo fractional differentiation operator.

The paper is outlined as follows. Preliminary results and motivation are discussed in Section 2. The numerical integration scheme is described and validated by comparing the numerical solution with a known solution in Section 3. The integration scheme is applied to an FDE with no known analytical solution in Section 4. Concluding remarks are given in Section 5.

2. Preliminaries and Motivation

2.1. *The Generalized Differential Operator Scheme for ODEs*

The main point of the proposed numerical scheme for FDEs is based on the transformation of the considered FDE into a corresponding ODE [23]. The solutions to the obtained ODEs can be constructed via the generalized differential operator technique. A short outline of this technique is given in this section. An in-depth review for n-th-order differential equations is presented in [24], and for systems of differential equations in [25].

2.1.1. The Construction of Analytic Solutions to ODEs in the Series Form

Consider the following explicit n-th-order ODE initial value problem with respect to function $z = z(x)$:

$$\frac{\mathrm{d}^n z}{\mathrm{d}x^n} = P\left(x, \frac{\mathrm{d}z}{\mathrm{d}x}, \ldots, \frac{\mathrm{d}^{n-1}z}{\mathrm{d}x^{n-1}}\right); \tag{2}$$

$$z(c) = s_0; \quad \left.\frac{\mathrm{d}^k z}{\mathrm{d}x^k}\right|_{x=c} = s_k; \quad k = 1, \ldots, n-1. \tag{3}$$

The generalized differential operator respective to (2), (3) reads:

$$\mathbf{D} = \mathbf{D}_c + \sum_{k=0}^{n-2} s_{k+1}\mathbf{D}_{s_k} + P(c, s_0, \ldots, s_{n-1})\mathbf{D}_{s_{n-1}}, \tag{4}$$

where $\mathbf{D}_\alpha$ denotes the partial differentiation operator with respect to variable α.

Let

$$p_j = p_j(c, s_0, \ldots, s_{n-1}) = \mathbf{D}^j s_0. \tag{5}$$

The series solution to (2), (3) reads:

$$z = \sum_{j=0}^{+\infty} \frac{(x-c)^j}{j!} p_j(c, s_0, \ldots, s_{n-1}). \tag{6}$$

2.1.2. The Construction of Closed-Form Solutions to ODEs

Necessary and sufficient conditions when the analytic series solution can be transformed into a closed-form solution are given in [24] and are briefly described in this sub-section.

Let us define $q_j = \frac{p_j}{j!}; j = 0, 1, \ldots$ and consider the following sequence of Hankel determinants:

$$d_n = \det\left(q_{j+k-2}\right)_{1 \leqslant j,k \leqslant n+1}; \quad n = 1, 2, \ldots \tag{7}$$

The sequence of series coefficients $q_j, j = 0, 1, \ldots$ is an m-th-order linear recurring sequence if and only if the following conditions hold true for the sequence of Hankel determinants [26]:

$$d_m \neq 0; \quad d_{m+k} = 0, \quad k = 1, 2, \ldots; \quad m \in \mathbb{N}. \tag{8}$$

If the above conditions do hold true, the coefficients q_j can be expressed as:

$$q_j = \sum_{k=1}^{m} \lambda_k \rho_k^j, \tag{9}$$

where λ_k are constants and ρ_k are roots of the characteristic polynomial [26]:

$$\begin{vmatrix} q_0 & q_1 & \cdots & q_m \\ q_1 & q_2 & \cdots & q_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ q_{m-1} & q_m & \cdots & q_{2m-1} \\ 1 & \rho & \cdots & \rho^m \end{vmatrix} = 0. \tag{10}$$

Combining the series solutions (6) and (10) and using the geometric progression sum formula $\sum_{j=0}^{+\infty} q^j = \frac{1}{1-q}, |q| < 1$, the solution to (2), (3) can be expressed in the closed form:

$$z = \sum_{j=0}^{+\infty} (x-c)^j q_j = \sum_{k=1}^{m} \lambda_k \sum_{j=0}^{+\infty} \rho_k^j (x-c)^j = \sum_{k=1}^{m} \frac{\lambda_k}{1 - \rho_k (x-c)}. \tag{11}$$

As shown in [24], (11) can be transformed into a solitary solution with a particular substitution. However, this transformation can be performed if and only if the sequence of coefficients $\lambda_k, k = 0, 1, \ldots$ is a linear recurring sequence.

2.1.3. Truncated Series and Shifted Centers of the Expansion

The solution to a given ODE can be approximated by a truncated series when it is not possible to transform the series solution into the closed form. Note that this is a straightforward operation since the analytic expressions of the coefficients $p_j(c, s_0, \ldots, s_{n-1})$ can be produced by the generalized differential operator.

Consider a first-order ODE:

$$z' = P(x, z); \quad z(c) = s. \tag{12}$$

The derivations described in previous sections yield the series solution to (12):

$$z(x, c, s) = \sum_{j=0}^{+\infty} \frac{(x-c)^j}{j!} p_j(c, s). \tag{13}$$

Let us set $c_0 = c, s_0 = s$ and consider $z_N(x, c_0, s_0)$ a truncated power series (13) by limiting the highest-order terms to $x^N; N \in \mathbb{N}$:

$$z_N(x, c_0, s_0) = \sum_{j=0}^{N} \frac{(x-c_0)^j}{j!} p_j(c_0, s_0). \tag{14}$$

Naturally, (14) is an approximation to (13) and generally decreases in accuracy as x moves further away from the expansion center c_0. However, a new approximation $z_N(x, c_1, s_1)$

at center $x = c_1 = c_0 + h_1$ can be derived from (13). The parameter s_1 is chosen as $s_1 = z_N(c_1, c_0, s_0)$ in order to ensure that $z_N(c_1, c_0, s_0) = z_N(c_1, c_1, s_1)$. The described steps can be repeated for a new center, yielding a piecewise-polynomial approximation $\widehat{z}(x)$ of the solution to (12):

$$\widehat{z}(x) = z_N(x, c_k, s_k), \ c_k \leqslant x < c_{k+1}, \quad k = 0, 1, \dots, \tag{15}$$

where $c_0 = c < c_1 < \dots < c_k < \dots$ and $s_0 = s, s_k = z_N(c_k, c_{k-1}, s_{k-1})$. The difference $h_k = c_k - c_{k-1} > 0$ is denoted as the step-size of the k-th step. The selection of this step-size to maintain a chosen level of error between the real solution $z(x)$ and the approximation $\widehat{z}(x)$ is a non-trivial problem, which is considered in the remainder of the paper.

2.2. The Ordinary Riccati Equation and Its Solution

As this paper deals with the fractional Riccati Equation (1), it is important to state the main results concerning its ordinary counterpart.

Consider the Riccati differential equation [27]:

$$\frac{\mathrm{d}z}{\mathrm{d}x} = A_0 + A_1 z + A_2 z^2; \quad z(c) = s. \tag{16}$$

It is well-known that this equation admits kink solitary solutions [25,27]. However, they cannot be directly obtained using the generalized differential operator technique. The generalized differential operator with respect to (16) reads:

$$\mathbf{D} = \mathbf{D}_c + \left(A_0 + A_1 s + A_2 s^2\right)\mathbf{D}_s. \tag{17}$$

The solution to (16) is then given by (6).

Let $p_j = \mathbf{D}^j s$ and define the sequence of coefficients $\frac{p_j}{j!}; j = 0, 1, \dots$. Because this sequence does not satisfy the condition (8) for any m, the sequence is not a linear recurring sequence and the solution to the Riccati equation cannot be constructed using the algorithm described above. However, the following independent variable substitution

$$\widehat{x} = \exp(\eta x); \quad \widehat{z}(\widehat{x}) = z\left(\frac{1}{\eta} \ln \widehat{x}\right) = z(x), \tag{18}$$

where $\eta \in \mathbb{R}, \eta \neq 0$, yields the transformed Riccati equation:

$$\eta \widehat{x} \frac{\mathrm{d}\widehat{z}}{\mathrm{d}\widehat{x}} = A_0 + A_1 \widehat{z} + A_2 \widehat{z}^2; \quad \widehat{z}(\widehat{c}) = s, \tag{19}$$

where $\widehat{c} = \exp(\eta c)$.

The generalized differential operator for (19) reads:

$$\widehat{\mathbf{D}} = \mathbf{D}_{\widehat{c}} + \frac{1}{\eta \widehat{c}}\left(A_0 + A_1 s + A_2 s^2\right)\mathbf{D}_s. \tag{20}$$

Defining $\widehat{p}_j = \widehat{\mathbf{D}}^j s; j = 0, 1, \dots$ now yields the sequence $\widehat{q}_j = \frac{\widehat{p}_j}{j!}$, which becomes a linear recurring sequence of order 2 at $\eta = A_2(z_1 - z_2)$, where z_1, z_2 are roots of the polynomial $A_2 z^2 + A_1 z + A_0 = 0$ [28].

This result yields the closed-form kink solitary solution to the Riccati equation [28]:

$$z(x) = \frac{z_2(s - z_1)\exp\left(\eta(x - c)\right) - y_1(s - y_2)}{(s - z_1)\exp\left(\eta(x - c)\right) - (s - y_2)} \tag{21}$$

2.3. The Fractional Power Series and Caputo Differentiation

Analytic solutions to fractional differential equations can be represented in the form of the fractional power series [23]:

$$f(x) = \sum_{j=0}^{+\infty} v_j \omega_j^{(n)}(x), \tag{22}$$

where $n \in \mathbb{N}$ denotes the order of the basis of fractional power series, $v_j \in \mathbb{R}$ are the coefficients of the series, and $\omega_j^{(n)}$ are the base functions defined as follows [23]:

$$\omega_j^{(n)}(x) = \frac{x^{\frac{j}{n}}}{\Gamma\left(\frac{j}{n}+1\right)}; \quad n \in \mathbb{N}; \quad j = 0, 1, \ldots, \tag{23}$$

where $\Gamma(x)$ denotes the Gamma function [29].

The Caputo differentiation operator ${}^{C}\mathbf{D}^{1/n}$ is defined for the base functions [30]:

$${}^{C}\mathbf{D}^{1/n}\omega_0^{(n)}(x) = 0; \quad {}^{C}\mathbf{D}^{1/n}\omega_j^{(n)}(x) = \omega_{j-1}^{(n)}, j = 1, 2, \ldots \tag{24}$$

Subsequently, the order k/n Caputo derivative of (22) reads:

$$\left({}^{C}\mathbf{D}^{1/n}\right)^k f(x) = \sum_{j=0}^{+\infty} v_{j+k}\omega_j^{(n)}(x). \tag{25}$$

Note that this definition of the Caputo differentiation operator is congruent with the classical integral-based definition [31].

2.4. Motivation: The Fractional Riccati Equation

Note that the closed-form solution to the Riccati ODE (16) cannot be constructed directly using the generalized differential operator technique. The substitution (18) is needed to map the Riccati ODE to (19), which in turn can be solved via the method described in the previous section.

Due to these reasons, we consider the Riccati fractional differential equation in the form (1) as a generalization of (19) rather than directly considering the fractional analogue of the Riccati Equation (16). As shown in [32], closed-form solutions to equations of the form (1) can be constructed, which is of vital importance to assess the efficacy of the numerical scheme presented in this paper.

2.5. Transformation of the FDE into the Characteristic ODE

Consider the following fractional differential equation:

$$\left({}^{C}\mathbf{D}^{1/n}\right)^n y = Q(y); \tag{26}$$

$$y(x_0) = u_0; \quad \left({}^{C}\mathbf{D}^{1/n}\right)^k y\Big|_{x=0} = v_k, \quad k = 1, 2, \ldots, n-1. \tag{27}$$

As demonstrated in detail in [23], setting $\hat{y} = y(t^n)$ and rearranging transforms (26) into the following ODE:

$$\hat{y}'_t = n\left(t^{n-1}Q(\hat{y}) + \sum_{j=1}^{n-1} \frac{v_j}{\Gamma\left(\frac{j}{n}+1\right)} t^{j-1}\right); \quad \hat{y}\left(\sqrt[n]{x_0}\right) = u_0. \tag{28}$$

The analytic solution to (28) yields the solution to (26) [23]:

$$y(x) = \hat{y}\left(\sqrt[n]{x}\right). \tag{29}$$

Thus, (26) and (28) are equivalent: if an analytical or numerical solution to (28) can be constructed, it immediately yields the FDE solution via (29).

3. The Development of the Numerical FDE Integration Scheme

3.1. Adaptive Step-Size Selection Strategy for the FDE Integration Scheme

As discussed in Section 2.1.3, the development of the step-size selection strategy for the numerical FDE integration scheme is necessary to ensure a chosen level of computational errors between the real and the approximated solutions. A fractional differential equation with the known analytic closed-form solution is investigated in this section.

Consider the following fractional Riccati equation:

$$x\left({}^{C}\mathbf{D}^{1/2}\right)^{2} y = 1 - 2y + y^{2}; \tag{30}$$

$$y(1) = 1; \quad {}^{C}\mathbf{D}^{1/2}y\Big|_{x=0} = -1. \tag{31}$$

Transforming (30)–(31) into the characteristic ODE (see Section 2.5) yields:

$$\frac{\mathrm{d}\hat{y}}{\mathrm{d}x} = \frac{2\left(1 - 2\hat{y} + \hat{y}^{2}\right)}{x} - \frac{2}{\sqrt{\pi}}; \tag{32}$$

$$\hat{y}(1) = 1; \quad \hat{y} = \hat{y}(x); \quad \hat{y}(\sqrt{x}) = y(x). \tag{33}$$

The initial-value problem (32)–(33) has the following analytic closed-form solution [32]:

$$\hat{y}(x) = \frac{\gamma_1\left(Y_1(\gamma_1)J_1(\gamma_2) - J_1(\gamma_1)Y_1(\gamma_2)\right)}{4\left(Y_0(\gamma_1)J_1(\gamma_2) - J_0(\gamma_1)Y_1(\gamma_2)\right)} + 1, \tag{34}$$

where $\gamma_1 = 4\sqrt{-\frac{x}{\sqrt{\pi}}}$, $\gamma_2 = 4\sqrt{-\frac{1}{\sqrt{\pi}}}$; $J_\beta(x)$ and $Y_\beta(x)$ are Bessel functions of the first and second kind respectively.

Alternatively, the numerical solution to (32)–(33) can be obtained via the technique presented in Section 2.1.3. Let $\hat{y}_N(x,c,s)$ denote the truncated power series approximation to (32)–(33):

$$\hat{y}_N(x,c,s) = \sum_{j=0}^{N} \frac{(x-c)^j}{j!} p_j(c,s), \quad N \in \mathbb{N}. \tag{35}$$

The analytical expressions of coefficients $p_j(c,s)$, $j = 0,\ldots,7$ are given in Appendix A.

Let us execute the following steps:

- Step 1. Let $c_0 = s_0 = 1$. The absolute differences $\Delta_N(x,c_0,s_0) = |\hat{y}(x) - \hat{y}_N(x,c_0,s_0)|$ are computed for $N = 0,\ldots,10$ and $x \in [1,L]$, where L is the upper bound of x. The contour plot depicting various levels of $\Delta_N(x,c_0,s_0)$ is presented in Figure 1a. It can be observed that for a fixed value of N the value of $\Delta_N(x,c_0,s_0)$ increases as x increases. New initial values c_1, s_1 for the next approximation are computed as follows:

$$c_1 = \arg\max_x \Delta_N(x,c_0,s_0) \leqslant \delta; \quad s_1 = \hat{y}_N(c_1,c_0,s_0), \tag{36}$$

where δ is the maximal allowed error of the numerical solution. Naturally, higher values of N result in larger values of c_1 at a cost of greater computation time. Let $N = 6$. Then, the resulting values of c_1 for different levels of δ are displayed in Figure 1b (denoted as black dashed lines, while the thick gray line denotes the analytical solution and the black solid line denotes the numerical solution). Figure 1 part (a) contains a contour plot with various levels of $\Delta_N(x, c_0, s_0)$ (absolute differences between the exact and numerical solutions to (32)–(33)). Parameter δ is set to 10^{-5} for the remainder of this computational experiment, as shown by the point in Figure 1a.

- Step $k = 2, 3, \ldots, K$. Analogous computations are performed for steps $k = 2, 3, \ldots, K$. Firstly, differences $\Delta_N(x, c_{k-1}, s_{k-1}) = |\widehat{y}(x) - \widehat{y}_N(x, c_{k-1}, s_{k-1})|$ are evaluated for $N = 0, \ldots, 10$ and $x \in [c_{k-1}, L]$ and then new initial values c_k, s_k are computed:

$$c_k = \arg\max_x \Delta_N(x, c_{k-1}, s_{k-1}) \leqslant \delta; \quad s_1 = \widehat{y}_N(c_k, c_{k-1}, s_{k-1}). \tag{37}$$

Results of the steps $k = 2, 3$ are displayed in Figures 2 and 3.

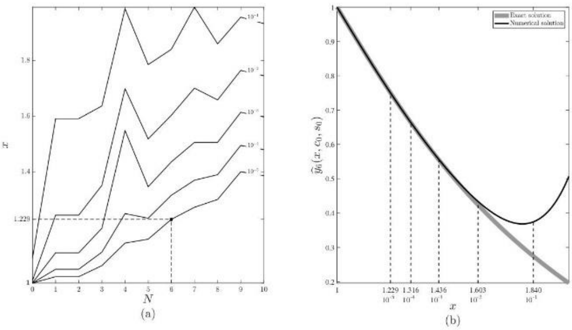


Figure 1. The determination of the second set of initial values for the numerical solution (35). The first set of initial values is $c_0 = 1$, $s_0 = 1$. Part (**a**) depicts a contour plot of the error for different values of N. Part (**b**) depicts the next initial points for different errors for $N = 6$.

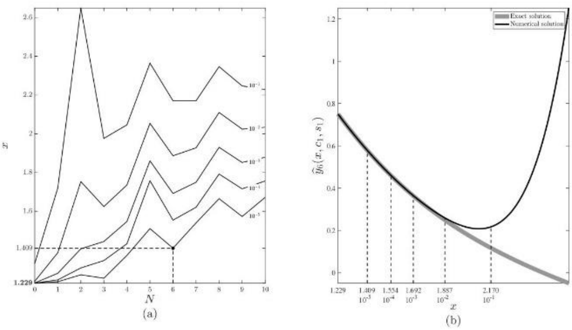


Figure 2. The determination of the third set of initial values for the numerical solution (35). The second set of initial values is $c_1 = 1.229$, $s_1 = 0.750$. Part (**a**) depicts a contour plot of the error for different values of N. Part (**b**) depicts the next initial points for different errors for $N = 6$.

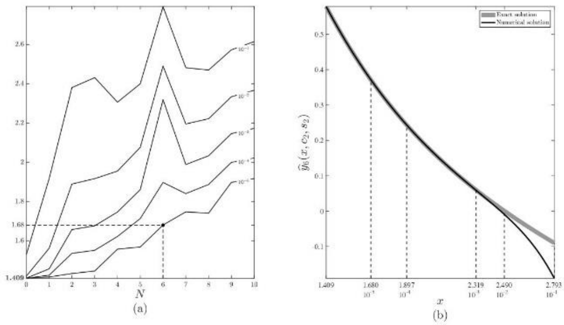


Figure 3. The determination of the fourth set of initial values for the numerical solution (35). The third set of initial values is $c_2 = 1.409$, $s_2 = 0.578$. Part (**a**) depicts a contour plot of the error for different values of N. Part (**b**) depicts the next initial points for different errors for $N = 6$.

The number of steps K is set to $K = 6$ in this study. The final piecewise-polynomial approximation $\widehat{y}_N(x)$ to (32)–(33) is depicted in Figure 4. Let the change in the numerical solution $\widehat{y}_N(x)$ at each step be denoted as

$$\Delta\widehat{y}_N^{(k)} = \max_{c_{k-1} \leqslant x \leqslant c_k} \widehat{y}_N(x, c_{k-1}, s_{k-1}) - \min_{c_{k-1} \leqslant x \leqslant c_k} \widehat{y}_N(x, c_{k-1}, s_{k-1}); \quad k = 1, \ldots, K. \tag{38}$$

The relationship between $\Delta\widehat{y}_N^{(k)}$ and the step-size h_k can be approximated via the following linear regression line:

$$\Delta\widehat{y}_N = \kappa_0^{(N)} + \kappa_1^{(N)} h, \tag{39}$$

where $\kappa_0^{(N)}, \kappa_1^{(N)} \in \mathbb{R}$ are regression coefficients. The constructions of linear regressions for $N = 6$ and $N = 7$ are illustrated in Figure 5 (parts (a) and (b), respectively). Black circles depict the values obtained from the final piecewise-polynomial approximation shown in Figure 4. The digits above the black circles denote the step number. The gray line corresponds to the linear regression (39). Regression equations for $N = 6$ (part (a)) and $N = 7$ (part (b)) read $\Delta\widehat{y}_N = 0.26572 - 0.29293h$ and $\Delta\widehat{y}_N = 0.27996 - 0.11987h$, respectively.

All computation steps performed for $N = 6$ were also repeated for $N = 7$ to obtain the regression depicted in Figure 5b. Note that while the coefficient of h decreases for a higher-order approximation, the overall trend remains unchanged.

The identification of the relationship between the change in the numerical solution and the step-size can be incorporated into the numerical FDE integration scheme that is described in the next section.

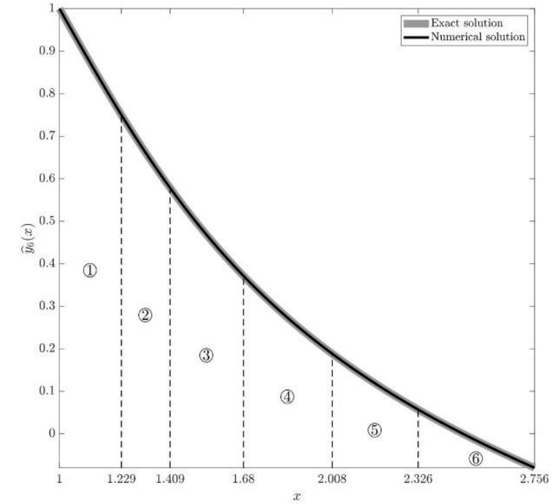


Figure 4. Gray and black solid lines correspond to the exact solution and the piecewise-polynomial approximation to (32)–(33), respectively ($N = 6$, $\delta = 10^{-5}$). Black dashed lines separate the parts of the numerical solution obtained at different steps. Circled digits denote the step number.

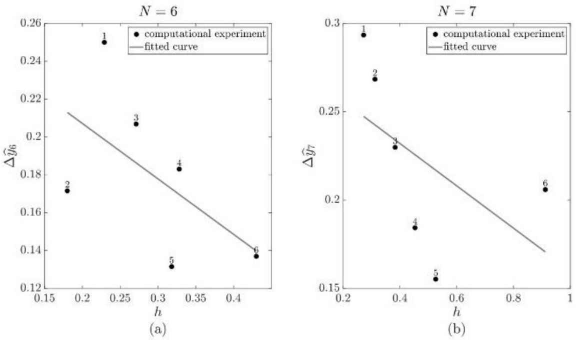


Figure 5. The relationship between $\Delta\widehat{y}_N$ (the change in the numerical solution $\widehat{y}_N(x)$) and the step-size h. Parts (**a**,**b**) correspond to $N = 6$ and $N = 7$, respectively.

3.2. The Implementation of the Numerical FDE Integration Scheme

Let us consider the following fractional differential equation:

$$\left({}^{C}\mathbf{D}^{1/2}\right)^{2} y = Q(y); \tag{40}$$

$$y(x_0) = u_0; \quad \left({}^{C}\mathbf{D}^{1/2}\right) y\Big|_{x=0} = v_0. \tag{41}$$

The numerical solution to (40)–(41) can be obtained via the integration scheme presented below:

1. Transform the FDE (40)–(41) into the characteristic ODE using the procedure described in Section 2.5:
$$\frac{\mathrm{d}\widehat{y}}{\mathrm{d}x} = P(\widehat{y}, v_0); \tag{42}$$
$$\widehat{y}(c_0) = s_0. \tag{43}$$
2. Obtain analytic expressions of coefficients $p_j(c, s)$ in the series solution (35) to the ODE (42)–(43) (see Section 2.1.1).
3. Fix the values of the following parameters: the order of the approximation N, the upper bound of the independent variable L, the upper bound for the step-size $h^{(U)}$, the upper bound for the change in the numerical solution $\Delta\widehat{y}_N^{(U)}$. Note that the recommended values for the parameters $h^{(U)}$ and $\Delta\widehat{y}_N^{(U)}$ are derived from the study presented in the previous section (Figure 5). The value $h^{(U)}$ corresponds to the highest value of h on the regression line, while the value $\Delta\widehat{y}_N^{(U)}$ corresponds to the highest value of $\Delta\widehat{y}_N$ on the regression line.
4. Repeat the following steps until the upper bound L is reached ($k = 0, 1, 2\ldots$):
   - Evaluate coefficients $p_j(c_k, s_k), j = 1, \ldots, N$.
   - Find the lowest value of x at which at least one of the following conditions is violated:
   $$h_k(x) = x - c_k \leqslant h^{(U)}; \tag{44}$$
   $$\Delta\widehat{y}_N^{(k)}(x) = \max_{c_k \leqslant \tilde{x} \leqslant x} \widehat{y}_N(\tilde{x}, c_k, s_k) - \max_{c_k \leqslant \tilde{x} \leqslant x} \widehat{y}_N(\tilde{x}, c_k, s_k) \leqslant \Delta\widehat{y}_N^{(U)}; \tag{45}$$
   $$\Delta\widehat{y}_N^{(k)}(x) \leqslant \kappa_0^{(N)} + \kappa_1^{(N)} h_k(x), \tag{46}$$
   where $\kappa_0^{(N)}, \kappa_1^{(N)} \in \mathbb{R}$ are regression coefficients determined in Section 3.1. The maximum and minimum values in (45) are necessary to ensure that the change in the numerical solution is computed correctly for non-monotonous and periodic functions.
   - Assign new initial values:
   $$c_{k+1} = x - \varepsilon; \quad s_{k+1} = \widehat{y}_N(c_{k+1}, c_k, s_k), \tag{47}$$
   where ε is an arbitrary small number.
5. Combine the obtained parts of the numerical solution to the ODE (42)–(43) into the piecewise-polynomial approximation $\widehat{y}_N(x)$:
$$\widehat{y}_N(x) = \widehat{y}_N(x, c_k, s_k), \; c_k \leqslant x < c_{k+1}, \quad k = 0, 1, \ldots \tag{48}$$
6. Construct the numerical solution to the FDE (40)–(41) by applying $y_N(x) = \widehat{y}_N\left(\sqrt{x}\right)$.

In order to validate the proposed numerical FDE integration scheme, it is applied to the FDE (30)–(31) presented in the previous section. The resulting piecewise-polynomial approximations $\hat{y}_N(x)$ and $y_N(x)$ are depicted in Figure 6. Part (a) depicts the exact (gray solid line) and the numerical (black solid line) solutions to the characteristic ODE (32)–(33) ($N = 6$, $L = 3$, $\delta = 10^{-5}$). Black dashed lines separate the parts of the numerical solution obtained at different steps. Circled digits denote the step number. Part (b) displays the exact solution (solid gray line) and piecewise-polynomial approximation (black solid line) to the initial FDE (30)–(31).

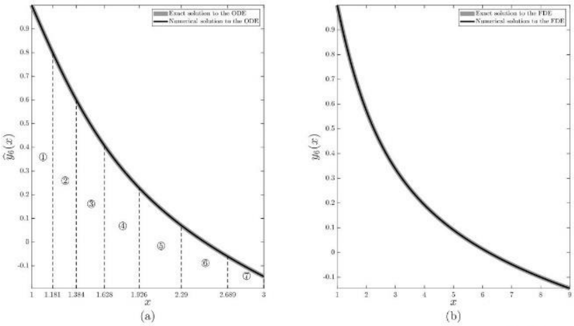


Figure 6. The application of the numerical FDE integration scheme to (30)–(31). Part (**a**) depicts the exact and approximate solutions to the characteristic ODE, while part (**b**) depicts the exact and approximate solutions to the FDE.

4. The Application of the Proposed Numerical FDE Integration Scheme

Consider the following fractional Riccati-type equation:

$$x\left({}^{C}\mathbf{D}^{1/2}\right)^{2}y = 1 - 2y + y^2 - y^3; \tag{49}$$

$$y(1) = 1; \quad {}^{C}\mathbf{D}^{1/2}y\Big|_{x=0} = -1. \tag{50}$$

Transforming (49)–(50) into the characteristic ODE (see Appendix B for a detailed derivation) yields:

$$\frac{\mathrm{d}\hat{y}}{\mathrm{d}x} = \frac{2\left(1 - 2\hat{y} + \hat{y}^2 - \hat{y}^3\right)}{x} - \frac{2}{\sqrt{\pi}}; \tag{51}$$

$$\hat{y}(1) = 1; \quad \hat{y} = \hat{y}(x); \quad \hat{y}(\sqrt{x}) = y(x). \tag{52}$$

Note that the FDE (49)–(50) does have a solution (the existence of the solution follows from (26)–(28)). However, the solution to (51)–(52) cannot be expressed in a closed form, because the coefficients p_j, as defined in (5), do not form a linear recurring sequence. Furthermore, (51) cannot be transformed via such an independent variable substitution if the coefficients would form a linear recurring sequence. The analytical expressions of coefficients $p_j(c,s), j = 0,\ldots,6$ can be found in Appendix C.

The numerical FDE integration scheme presented in the previous section is used in order to obtain the piecewise-polynomial approximations $\hat{y}_N(x)$ and $y_N(x)$ to (51)–(52) and (49)–(50), respectively. The linear regression equation approximating the relationship between the step-size h_k and the change in the numerical solution $\Delta\hat{y}_N^{(k)}$ derived in Section 3.1 for $N = 6$ (Figure 5 part (a)) is used in order to adaptively select the optimal step-size. The resulting numerical solutions $\hat{y}_N(x)$ and $y_N(x)$ are displayed in the Figure 7. Part (a) depicts the numerical solution to the characteristic ODE (51)–(52) ($N = 6$, $L = 3$, $\delta = 10^{-5}$).

Black dashed lines separate the parts of the numerical solution obtained at different steps. Circled digits denote the step number. Part (b) displays the piecewise-polynomial approximation to the initial FDE (49)–(50). The values of h_k and $\Delta\hat{y}_N^{(k)}$ at each step are presented in Table 1.

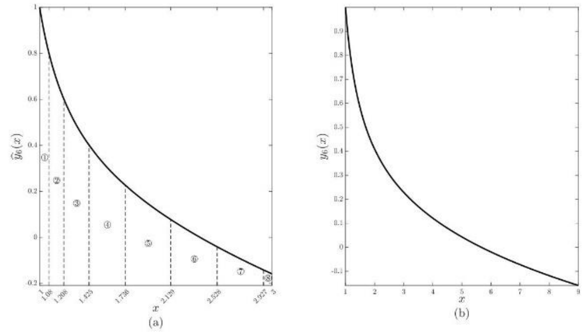


Figure 7. The application of the numerical FDE integration scheme to (49)–(50). Part (**a**) depicts the exact and approximate solutions to the characteristic ODE, while part (**b**) depicts the exact and approximate solutions to the FDE.

Table 1. The values of the step-size h_k and the variation $\Delta\hat{y}_N^{(k)}$ in the numerical solution to (51)–(52) at each step $k = 1, \ldots, 8$.

Step k	h_k	$\Delta\hat{y}_6^{(k)}$
1	0.080	0.1992
2	0.128	0.1990
3	0.217	0.2000
4	0.311	0.1743
5	0.393	0.1500
6	0.399	0.1185
7	0.399	0.1006
8	0.073	0.0170

5. Concluding Remarks

A novel semi-analytical scheme for the numerical integration of fractional differential equations was presented in this paper. The proposed integration scheme is adaptive: the approximation error can be selected arbitrarily, and the algorithm is adapted by using a higher-order piecewise-polynomial approximation. Furthermore, the scheme can easily be extended into higher-order fractional differential equations, since the generalized differential operator technique is applicable to differential systems of any order.

All computational experiments in this paper were performed on fractional Riccati-type nonlinear differential equations. Riccati equations play the central role in nonlinear dynamics because solutions to ordinary Riccati equations do represent soliton-type solutions [28,33]. Without doubt, the proposed scheme can be used for numerical integration of any other class of fractional differential equations.

While the FDEs analyzed in this paper have all had a base derivative order of $\alpha = \frac{1}{2}$ for simplicity, the scheme remains valid for any fractional derivative base order $\alpha = \frac{1}{k}$. This change is implemented by replacing the operator ${}^{C}\mathbf{D}^{1/2}$ with ${}^{C}\mathbf{D}^{1/k}$, while the algorithm to obtain the characteristic ODE remains unchanged except for a higher number of initial conditions.

The presented integration scheme cannot advance past a singularity point. The size of the integration step becomes arbitrarily small as the solution nears the singularity point. This fact can be considered the limitation of the scheme. However, this feature allows the description of the solution in the surrounding of the singularity point with a predefined accuracy.

The extension of the proposed integration scheme to singularity points remains a definite objective for future research. Since the presented scheme is semi-analytical, there are possibilities to adapt it in such a way that singularity points could be detected automatically. Other avenues of future works include adapting the scheme so that any numerical integration technique could be used while solving the characteristic ODE. While this would make the scheme purely numerical and pose challenges in changing the timescale (since the approximation would no longer be a polynomial function), it could potentially open up new possibilities in applying already existing results.

Author Contributions: Conceptualization and theorems, Z.N. and M.R.; methodology and proofs, Z.N., I.T. and T.T.; numerical experiments and computer algebra, I.T. and R.M.; writing—original draft preparation, I.T.; writing—review and editing, T.T. and M.R. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Acknowledgments: Technical support from the School of Mathematics and Natural Sciences of Kaunas University of Technology is acknowledged.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Analytical Expressions of the Coefficients $p_j(c,s)$ for the ODE (30)–(31)

Coefficients $p_j(c,s)$ in the series solution (35) to the ODE (30)–(31) read:

$$p_0(c,s) = s;$$

$$p_1(c,s) = -2\,\frac{-(s-1)^2\sqrt{\pi}+c}{c\sqrt{\pi}};$$

$$p_2(c,s) = -8\,\frac{\left(\left(-s^2+9/4\,s-5/4\right)\sqrt{\pi}+c\right)(s-1)}{\sqrt{\pi}c^2};$$

$$p_3(c,s) = \frac{-64\,(s-5/4)c(s-1)\sqrt{\pi}+\left(48\,s^4-216\,s^3+364\,s^2-272\,s+76\right)\pi+16\,c^2}{c^3\pi};$$

$$p_4(c,s) = 256\,\frac{1}{\pi^{3/2}c^4}\left(3/2\left(s^3-\frac{15\,s^2}{4}+\frac{227\,s}{48}-\frac{193}{96}\right)(s-1)^2\pi^{3/2}+\right.$$

$$\left.+\,c\left(c\left(s-\frac{19}{16}\right)\sqrt{\pi}-5/2\,\pi\left(s^2-\frac{99\,s}{40}+\frac{31}{20}\right)(s-1)\right)\right);$$

$$p_5(c,s) = \frac{1}{\pi^2 c^5}\left(-7680\,(s-4/3)\left(s^2 - \frac{19\,s}{8} + \frac{29}{20}\right)c(s-1)\pi^{3/2} - 512\,\sqrt{\pi}c^3 + 3840\,\pi^2 s^6 - \right.$$
$$-26880\,\pi^2 s^5 + 78480\,\pi^2 s^4 - 122320\,\pi^2 s^3 + \left(4352\,\pi\,c^2 + 107328\,\pi^2\right)s^2 + \left(-10432\,\pi\,c^2 - 50256\,\pi^2\right)s +$$
$$\left. +6272\,\pi\,c^2 + 9808\,\pi^2\right);$$
$$p_6(c,s) = -17408\,\frac{1}{\pi^{3/2}c^6}\left(-\frac{45\,(s-1)^2\pi^{3/2}}{17}\left(s^5 - \frac{25\,s^4}{4} + \frac{377\,s^3}{24} - \frac{1271\,s^2}{64} + \frac{36407\,s}{2880} - \frac{9347}{2880}\right) + \right.$$
$$+\left(-\frac{77\,c\sqrt{\pi}}{17}\left(s^3 - \frac{279\,s^2}{77} + \frac{5409\,s}{1232} - \frac{137}{77}\right) + \frac{105\,\pi\,s^5}{17} - \frac{4995\,\pi\,s^4}{136} + \frac{11903\,\pi\,s^3}{136} - \frac{56835\,\pi\,s^2}{544} + \right.$$
$$\left.\left. + \left(c^2 + \frac{1062\,\pi}{17}\right)s - \frac{171\,c^2}{136} - \frac{8141\,\pi}{544}\right)c\right);$$
$$p_7(c,s) = \frac{1}{\pi^2 c^7}\left(-1720320\left(s^5 - \frac{99\,s^4}{16} + \frac{9843\,s^3}{640} - \frac{18437\,s^2}{960} + \frac{324067\,s}{26880} - \frac{81887}{26880}\right)c(s-1)\pi^{3/2} - \right.$$
$$-507904\left(s^2 - \frac{1235\,s}{496} + \frac{3097}{1984}\right)c^3\sqrt{\pi} + 645120\,\pi^2 s^8 - 6128640\,\pi^2 s^7 + 25509120\,\pi^2 s^6 - 60762240\,\pi^2 s^5 +$$
$$+\left(1548288\,\pi\,c^2 + 90596352\,\pi^2\right)s^4 + \left(-7519232\,\pi\,c^2 - 86583840\,\pi^2\right)s^3 + \left(13735296\,\pi\,c^2 + 51798528\,\pi^2\right)s^2 +$$
$$\left. + \left(-11187072\,\pi\,c^2 - 17734944\,\pi^2\right)s + 34816\,c^4 + 3428480\,\pi\,c^2 + 2660544\,\pi^2\right).$$

Appendix B. Transformation of FDE (49) into the Characteristic ODE (51)

Consider the FDE initial value problem (49), (50). The solution can be written in series form as:

$$y = \sum_{j=0}^{+\infty} \nu_j \omega_j^{(2)} = \sum_{j=0}^{+\infty} \gamma_j \left(\sqrt{x}\right)^j, \tag{A1}$$

where $\gamma_j = \frac{\nu_j}{\Gamma\left(\frac{j}{2}+1\right)}$. Note that the coefficients ν_0, ν_1 are given by the initial conditions (50), thus $\nu_0 = 1, \nu_1 = -1$.

Denote $Q(y) = 1 - 2y + y^2 - y^3$ for brevity. Inserting the series solution into (49) yields:

$$x \sum_{j=0}^{+\infty} \left(\frac{j}{2} + 1\right) \gamma_{j+2} \left(\sqrt{x}\right)^j = Q(y) \tag{A2}$$

Substituting $\sqrt{x} = t$ and rearranging results in:

$$\sum_{j=0}^{+\infty} (j+2)\gamma_{j+2} t^j = \frac{2}{t^2} Q(y). \tag{A3}$$

Multiplying both sides by t and re-indexing the left-hand side sum yields:

$$\sum_{j=2}^{+\infty} j\gamma_j t^{j-1} = \frac{2}{t} Q(y). \tag{A4}$$

Adding γ_1 to both sides results in:

$$\sum_{j=1}^{+\infty} j\gamma_j t^{j-1} = \frac{2}{t}Q(y) + \gamma_1. \tag{A5}$$

Let $\widehat{y}(t) = \sum_{j=0}^{+\infty} \gamma_j t^j$. Then, the left-hand side of (A5) is the derivative $\frac{\mathrm{d}\widehat{y}}{\mathrm{d}t}$, while the coefficient γ_1 reads $\gamma_1 = \frac{\nu_1}{\Gamma\left(\frac{3}{2}\right)} = -\frac{2}{\sqrt{\pi}}$. Combining these derivations yields:

$$\frac{\mathrm{d}\widehat{y}}{\mathrm{d}t} = \frac{2}{t}Q(y) - \frac{2}{\sqrt{\pi}}. \tag{A6}$$

The initial condition ${}^{C}\mathbf{D}^{1/2}y\Big|_{x=0} = -1$ has already been incorporated into the above equation. The other initial condition, $y(1) = 1$, is transformed into an equivalent condition $\widehat{y}(1) = 1$, since $\widehat{y}(\sqrt{x}) = y(x)$.

Appendix C. Analytical Expressions of the Coefficients $p_j(c,s)$ for the ODE (51)–(52)

Coefficients $p_j(c,s)$ in the series solution (35) to the ODE (51)–(52) read:

$$p_0(c,s) = s;$$

$$p_1(c,s) = -2\,\frac{\left(s^3 - s^2 + 2\,s - 1\right)\sqrt{\pi} + c}{c\sqrt{\pi}};$$

$$p_2(c,s) = 12\,\frac{\left(s^3 - s^2 + 2\,s - 1\right)\left(s^2 - 2/3\,s + 5/6\right)\sqrt{\pi} + c\left(s^2 - 2/3\,s + 2/3\right)}{\sqrt{\pi}c^2};$$

$$p_3(c,s) = \frac{1}{\pi\,c^3}\left(-168\,c\left(s^4 - 4/3\,s^3 + \frac{47\,s^2}{21} - \frac{10\,s}{7} + \frac{10}{21}\right)\sqrt{\pi} - 120\,\pi\,s^7 + 280\,\pi\,s^6 - 676\,\pi\,s^5 + \right.$$
$$\left. + 884\,\pi\,s^4 - 956\,\pi\,s^3 + 688\,\pi\,s^2 + \left(-48\,c^2 - 320\,\pi\right)s + 16\,c^2 + 76\,\pi\right);$$

$$p_4(c,s) = 96\,\frac{1}{\pi^{3/2}c^4}\left(\frac{\left(35\,s^3 - 35\,s^2 + 70\,s - 35\right)\pi^{3/2}}{2}\left(s^6 - 2\,s^5 + \frac{89\,s^4}{21} - \frac{472\,s^3}{105} + \frac{547\,s^2}{140} - \frac{407\,s}{210} + \frac{31}{60}\right) + \right.$$
$$+ c\left(15\,c\left(s^3 - s^2 + \frac{109\,s}{90} - \frac{37}{90}\right)\sqrt{\pi} + \frac{63\,\pi\,s^6}{2} - 63\,\pi\,s^5 + \frac{1579\,\pi\,s^4}{12} - 141\,\pi\,s^3 + \frac{1415\,\pi\,s^2}{12} + c^2 - \right.$$
$$\left.\left. \frac{178\,\pi\,s}{3} + \frac{40\,\pi}{3}\right)\right);$$

$$p_5(c,s) = \frac{1}{c^5\pi^2}\left(-66,528\,c\left(s^8 - 8/3\,s^7 + \frac{9103\,s^6}{1386} - \frac{19829\,s^5}{2079} + \frac{15503\,s^4}{1386} - \frac{2755\,s^3}{297} + \frac{3812\,s^2}{693} - \right.\right.$$

$$\left. + \frac{1468\,s}{693} + \frac{788}{2079}\right)\pi^{3/2} - 8640\,c^3\left(s^2 - 2/3\,s + \frac{56}{135}\right)\sqrt{\pi} - 44928\,\pi\left(\frac{35\,s^{11}\pi}{52} - \frac{385\,s^{10}\pi}{156} + \right.$$

$$+ \frac{10675\,s^9\pi}{1404} - \frac{10661\,s^8\pi}{702} + \frac{139693\,\pi\,s^7}{5616} - \frac{6543\,\pi\,s^6}{208} + \left(c^2 + \frac{180403\,\pi}{5616}\right)s^5 + \left(-5/3\,c^2 - \frac{72665\,\pi}{2808}\right)s^4 +$$

$$\left.\left. + \left(\frac{151\,c^2}{52} + \frac{5617\,\pi}{351}\right)s^3 + \left(-\frac{259\,c^2}{108} - \frac{13589\,\pi}{1872}\right)s^2 + \left(\frac{961\,c^2}{702} + \frac{997\,\pi}{468}\right)s - \frac{251\,c^2}{702} - \frac{841\,\pi}{2808}\right)\right);$$

$$p_6(c,s) = 34560\,\frac{1}{\pi^{5/2}c^6}\left(\frac{\left(77\,s^3 - 77\,s^2 + 154\,s - 77\right)\pi^{5/2}}{4}\left(s^{10} - 10/3\,s^9 + \frac{625\,s^8}{66} - \frac{25268\,s^7}{1485} + \right.\right.$$

$$\left. + \frac{13513\,s^6}{540} - \frac{21309\,s^5}{770} + \frac{404011\,s^4}{16632} - \frac{67087\,s^3}{4158} + \frac{72073\,s^2}{9240} - \frac{51517\,s}{20790} + \frac{16169}{41580}\right) +$$

$$+ c\left(\frac{219\,\pi^{3/2}c}{5}\left(s^7 - 7/3\,s^6 + \frac{1108\,s^5}{219} - \frac{12280\,s^4}{1971} + \frac{283223\,s^3}{47304} - \frac{20117\,s^2}{5256} + \frac{4037\,s}{2628} - \frac{7079}{23652}\right) + \right.$$

$$+ c^3(-1/3 + s)\sqrt{\pi} + 14\,\pi\left(\frac{143\,s^{10}\pi}{40} - \frac{143\,s^9\pi}{12} + \frac{1177\,s^8\pi}{35} - \frac{6347\,\pi\,s^7}{105} + \frac{2666387\,\pi\,s^6}{30240} - \frac{490493\,\pi\,s^5}{5040} + \right.$$

$$+ \left(c^2 + \frac{850561\,\pi}{10080}\right)s^4 + \left(-4/3\,c^2 - \frac{1944\,\pi}{35}\right)s^3 + \left(\frac{1529\,c^2}{840} + \frac{22031\,\pi}{840}\right)s^2 + \left(-\frac{3907\,c^2}{3780} - \frac{4509\,\pi}{560}\right)s +$$

$$\left.\left.\left. \frac{41\,c^2}{140} + \frac{8971\,\pi}{7560}\right)\right)\right).$$

References

1. Heymans, N. Fractional calculus description of non-linear viscoelastic behaviour of polymers. *Nonlinear Dyn.* **2004**, *38*, 221–231. [CrossRef]
2. Li, X.; Tian, X. Fractional order thermo-viscoelastic theory of biological tissue with dual phase lag heat conduction model. *Appl. Math. Model.* **2021**, *95*, 612–622. [CrossRef]
3. Dubey, V.P.; Dubey, S.; Kumar, D.; Singh, J. A computational study of fractional model of atmospheric dynamics of carbon dioxide gas. *Chaos Solitons Fractals* **2021**, *142*, 110375. [CrossRef]
4. Acay, B.; Inc, M. Fractional modeling of temperature dynamics of a building with singular kernels. *Chaos Solitons Fractals* **2021**, *142*, 110482. [CrossRef]
5. Chen, Y.; Liu, F.; Yu, Q.; Li, T. Review of fractional epidemic models. *Appl. Math. Model.* **2021**, *97*, 281–307. [CrossRef]
6. Biala, T.A.; Khaliq, A. A fractional-order compartmental model for the spread of the COVID-19 pandemic. *Commun. Nonlinear Sci. Numer. Simul.* **2021**, *98*, 105764. [CrossRef] [PubMed]
7. Rezaei, M.; Yazdanian, A.; Ashrafi, A.; Mahmoudi, S. Numerical pricing based on fractional Black–Scholes equation with time-dependent parameters under the CEV model: Double barrier options. *Comput. Math. Appl.* **2021**, *90*, 104–111. [CrossRef]
8. Tarasov, V.E. Fractional econophysics: Market price dynamics with memory effects. *Phys. A Stat. Mech. Appl.* **2020**, *557*, 124865. [CrossRef]
9. Betancur-Herrera, D.E.; Muñoz-Galeano, N. A numerical method for solving Caputo's and Riemann-Liouville's fractional differential equations which includes multi-order fractional derivatives and variable coefficients. *Commun. Nonlinear Sci. Numer. Simul.* **2020**, *84*, 105180. [CrossRef]
10. Alchikh, R.; Khuri, S. Numerical solution of a fractional differential equation arising in optics. *Optik* **2020**, *208*, 163911. [CrossRef]
11. Mendes, E.M.; Salgado, G.H.; Aguirre, L.A. Numerical solution of Caputo fractional differential equations with infinity memory effect at initial condition. *Commun. Nonlinear Sci. Numer. Simul.* **2019**, *69*, 237–247. [CrossRef]

12. Firoozjaee, M.; Jafari, H.; Lia, A.; Baleanu, D. Numerical approach of Fokker–Planck equation with Caputo–Fabrizio fractional derivative using Ritz approximation. *J. Comput. Appl. Math.* **2018**, *339*, 367–373. [CrossRef]
13. Kheybari, S. Numerical algorithm to Caputo type time–space fractional partial differential equations with variable coefficients. *Math. Comput. Simul.* **2021**, *182*, 66–85. [CrossRef]
14. Esmaeili, S.; Shamsi, M.; Luchko, Y. Numerical solution of fractional differential equations with a collocation method based on Müntz polynomials. *Comput. Math. Appl.* **2011**, *62*, 918–929. [CrossRef]
15. Han, W.; Chen, Y.M.; Liu, D.Y.; Li, X.L.; Boutat, D. Numerical solution for a class of multi-order fractional differential equations with error correction and convergence analysis. *Adv. Differ. Equ.* **2018**, *2018*, 1–22. [CrossRef]
16. Hinze, M.; Schmidt, A.; Leine, R.I. Numerical solution of fractional-order ordinary differential equations using the reformulated infinite state representation. *Fract. Calc. Appl. Anal.* **2019**, *22*, 1321–1350. [CrossRef]
17. Maleknejad, K.; Rashidinia, J.; Eftekhari, T. Numerical solutions of distributed order fractional differential equations in the time domain using the Müntz–Legendre wavelets approach. *Numer. Methods Partial Differ. Equ.* **2021**, *37*, 707–731. [CrossRef]
18. Babaaghaie, A.; Maleknejad, K. Numerical solutions of nonlinear two-dimensional partial Volterra integro-differential equations by Haar wavelet. *J. Comput. Appl. Math.* **2017**, *317*, 643–651. [CrossRef]
19. Garrappa, R. Numerical solution of fractional differential equations: A survey and a software tutorial. *Mathematics* **2018**, *6*, 16. [CrossRef]
20. Jafari, H.; Tajadodi, H. He's variational iteration method for solving fractional Riccati differential equation. *Int. J. Differ. Equ.* **2010**, *2010*, 764738. [CrossRef]
21. Khan, N.A.; Ara, A.; Jamil, M. An efficient approach for solving the Riccati equation with fractional orders. *Comput. Math. Appl.* **2011**, *61*, 2683–2689. [CrossRef]
22. Gohar, M. Approximate Solution to Fractional Riccati Differential Equations. *Fractals* **2019**, *27*, 1950128. [CrossRef]
23. Timofejeva, I.; Navickas, Z.; Telksnys, T.; Marcinkevičius, R.; Yang, X.J.; Ragulskis, M.K. The extension of analytic solutions to FDEs to the negative half-line. *AIMS Math.* **2021**, *6*, 3257–3271. [CrossRef]
24. Navickas, Z.; Bikulciene, L.; Ragulskis, M. Generalization of Exp-function and other standard function methods. *Appl. Math. Comput.* **2010**, *216*, 2380–2393. [CrossRef]
25. Navickas, Z.; Marcinkevicius, R.; Telksnys, T.; Ragulskis, M. Existence of second order solitary solutions to Riccati differential equations coupled with a multiplicative term. *IMA J. Appl. Math.* **2016**, *81*, 1163–1190. [CrossRef]
26. Kurakin, V.; Kuzmin, A.; Mikhalev, A.; Nechaev, A. Linear recurring sequences over rings and modules. *J. Math. Sci.* **1995**, *76*, 2793–2915. [CrossRef]
27. Zaitsev, V.F.; Polyanin, A.D. *Handbook of Exact Solutions for Ordinary Differential Equations*; CRC Press: Boca Raton, FL, USA, 2002.
28. Navickas, Z.; Ragulskis, M. How far one can go with the Exp-function method? *Appl. Math. Comput.* **2009**, *211*, 522–530. [CrossRef]
29. Andrews, G.E.; Askey, R.; Roy, R. *Special Functions*; Number 71; Cambridge University Press: Cambridge, MA, USA, 1999.
30. Navickas, Z.; Telksnys, T.; Timofejeva, I.; Marcinkevičius, R.; Ragulskis, M. An operator-based approach for the construction of closed-form solutions to fractional differential equations. *Math. Model. Anal.* **2018**, *23*, 665–685. [CrossRef]
31. Zhou, Y.; Wang, J.; Zhang, L. *Basic Theory of Fractional Differential Equations*; World Scientific: Singapore, 2016.
32. Navickas, Z.; Telksnys, T.; Marcinkevicius, R.; Ragulskis, M. Operator-based approach for the construction of analytical soliton solutions to nonlinear fractional-order differential equations. *Chaos Solitons Fractals* **2017**, *104*, 625–634. [CrossRef]
33. Scott, A. *Encyclopedia of Nonlinear Science*; Routledge: Abingdon-on-Thames, UK, 2006.

Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(9): 16536–16554.
DOI: 10.3934/math.2022905
Received: 26 May 2022
Revised: 05 July 2022
Accepted: 06 July 2022
Published: 08 July 2022

Research article

The construction of solutions to ${}^{C}\boldsymbol{D}^{(1/n)}$ type FDEs via reduction to $\left({}^{C}\boldsymbol{D}^{(1/n)}\right)^{n}$ type FDEs

R. Marcinkevicius[1], **I. Telksniene**[2,*], **T. Telksnys**[2], **Z. Navickas**[2] **and M. Ragulskis**[2]

[1] Department of Software Engineering, Kaunas University of Technology, Studentu 50-415, Kaunas LT-51368, Lithuania

[2] Center for Nonlinear Systems, Kaunas University of Technology, Studentu 50-147, Kaunas LT-51368, Lithuania

* **Correspondence:** Email: inga.telksniene@ktu.lt; Tel: +37063081477.

Abstract: A scheme for the integration of ${}^{C}\boldsymbol{D}^{(1/n)}$-type fractional differential equations (FDEs) is presented in this paper. The approach is based on the expansion of solutions to FDEs via fractional power series. It is proven that ${}^{C}\boldsymbol{D}^{(1/n)}$-type FDEs can be transformed into equivalent $\left({}^{C}\boldsymbol{D}^{(1/n)}\right)^{n}$-type FDEs via operator calculus techniques. The efficacy of the scheme is demonstrated by integrating the fractional Riccati differential equation.

Keywords: fractional differential equation; operator calculus; fractional power series expansion
Mathematics Subject Classification: 34A08, 30B10, 65L99

1. Introduction

In the past few decades, fractional differential equations (FDEs) have gone from being a niche area of mathematical analysis to the forefront of mathematical modeling. Finding applications in a myriad of areas ranging from the classical FDEs in viscoelasticity [12], to more novel physical fields [9] and beyond to biology and medicine [13]. A review on more recent applications of fractional differential equations in a variety of research fields is given below.

One of the foremost fields of research to feature fractional derivatives in recent years is biomedicine. A type of fractional logistic differential equation used to model the COVID-19 pandemic is discussed in [4, 17]. Continuous glucose monitoring is analyzed via a fractional differential equation model constructed from a noisy time series in [5]. Fractional differential equations have been used in a scheme to detect tea moisture content that was introduced in [27]. The memory property of fractional derivatives is exploited to study a combined drug treatment for the

Human Immunodeficiency Virus (HIV) in [24]. The Gompertz law, used in many areas of biophysics, is generalized using fractional derivatives in [7]. An FDE model for the interaction of nutrient phytoplankton and its predator zooplankton is considered in [3].

Models of financial and economic processes have also recently featured fractional derivatives. A review of fractional differential equations used in economic growth models is given in [11]. Systems of FDEs are used in [30] to construct an indicator for the evaluation of economic development of a given region. The evolution of fractional-order chaotic financial systems is studied using the Adams-Bashforth-Moulton method in [28]. A financial crisis model represented by a system of fractional differential equations is analyzed in [18].

In physics and engineering, optics is a field where fractional differential equations find many uses. Semi-analytical solutions to the fractional Eikonal equation, a problem in optics, are constructed in [1]. The Caudrey-Dodd-Gibbon equation, used in laser optics, is analyzed in its fractional form in [23]. Optical soliton solutions to the conformable fractional Benjamin-Bona-Mahony equations are constructed in [31]. A fractional order model studying light distribution from the main fiber into other branch fibers in optical meta-materials is analyzed in [2].

Techniques for integrating FDEs can be classified into two large categories: numerical and analytical methods. Recently, there has been a surge of interest in numerical methods due to the increased reliance on FDE in fields of applied research. A review of classical methods is given in [6], while more recent algorithms are discussed in [14].

Analytical or semi-analytical techniques for the construction of solutions to FDEs have also experienced recent developments. The natural transform method was applied to contruct analytical solutions to a fractional oscillator in a resisting medium model in [10]. The Laplace-Adomian decomposition method is used to obtain the analytical solutions to a class of fractional-order dispersive partial differential equations in [20]. The same approach yields the solutions of fractional Zakharov-Kuznetsov equations in [21]. The q-homotopy analysis transform method is applied to solve a class of fractional diffusion equations in [22].

A particular class of techniques based on fractional power series has been presented in [15, 16, 26]. This approach considers the $\left({}^{C}\boldsymbol{D}^{(1/n)}\right)^{n}$-type fractional equation:

$$\left({}^{C}\boldsymbol{D}^{(1/n)}\right)^{n} y = F(x, y); \quad y = y(x), \tag{1.1}$$

where ${}^{C}\boldsymbol{D}^{(1/n)}$ denotes the Caputo derivative of order $\frac{1}{n}$ with respect to independent variable x; F is an analytic function. Note that in the operator sense, the expression $\left({}^{C}\boldsymbol{D}^{(1/n)}\right)^{n}$ is not equivalent to the integer-order derivative $\dfrac{\mathrm{d}}{\mathrm{d}x}$ – while the set of solutions to (1.1) does include solutions of the ordinary differential equation $y' = F(x, y)$, it is a much wider set [16].

It was demonstrated in [15] that (1.1) can be mapped to an equivalent ordinary differential equation (ODE) via the use of fractional power series. The solution to the obtained ODE can then be transformed into a solution to the original FDE (1.1). The main objective of this paper is to extend this approach to ${}^{C}\boldsymbol{D}^{(1/n)}$-type FDEs:

$${}^{C}\boldsymbol{D}^{(1/n)} y = G(x, y), \tag{1.2}$$

where $G(x, y)$ is an analytic function. It is demonstrated that FDE (1.2) can be transformed into (1.1) if specific conditions hold true, which can then be solved via the integration scheme presented in [25].

Note that while $\left({}^{C}\boldsymbol{D}^{(1/n)}\right)^{n}$-type equations (1.1) do not necessarily have a physical interpretation, they are a vital part of the scheme presented in this paper for solving ${}^{C}\boldsymbol{D}^{(1/n)}$-type FDEs (1.2), which have a wide range of physical applications [9].

The paper is organized as follows: Section 2 contains preliminary results; Section 3 contains main definitions and derivations that demonstrate the transformation of ${}^{C}\boldsymbol{D}^{(1/n)}$-type FDEs into $\left({}^{C}\boldsymbol{D}^{(1/n)}\right)^{n}$-type FDEs via the Riccati equation; Section 4 contains numerical experiments demonstrating the efficacy of the presented scheme.

2. Preliminaries

2.1. Fractional power series

In this paper, all functions $f(x)$ are represented via power series consisting of fractional-order powers of the independent variable. If a fractional derivative of order $\alpha = \frac{k}{n}; \gcd(k,n) = 1$ ($\gcd(k,n)$ denotes the greatest common divisor of integers k and n) is considered, then the series parameter is set to n:

$$f(x) = \sum_{j=0}^{+\infty} c_j x^{\frac{j}{n}}; \quad c_j \in \mathbb{R},\ n \in \mathbb{N}. \tag{2.1}$$

Series (2.1) is required to converge in the neighbourhood $0 \leq x < R, \quad R > 0$. The series can be rewritten for a more convenient approach with regards to the Caputo fractional derivative in the following form:

$$f(x) = \sum_{j=0}^{+\infty} v_j w_j^{(n)}; \quad n \in \mathbb{N}, \tag{2.2}$$

where $w_j^{(n)}$, $j = 0, 1, \ldots$ are the basis elements of series $f(x)$:

$$w_j^{(n)} = \frac{x^{\frac{j}{n}}}{\Gamma\left(1 + \frac{j}{n}\right)}. \tag{2.3}$$

The following equality relates coefficients c_j and v_j:

$$v_j = c_j \Gamma\left(1 + \frac{j}{n}\right), \quad j = 0, 1, \ldots. \tag{2.4}$$

As mentioned previously, the series (2.2) and all subsequent fractional power series are required to converge in the neighbourhood $0 \leq x < R,\ R > 0$.

Note that the substitution $t = x^{\frac{1}{n}}$ can be used to convert (2.1) (and (2.2)) into an integer-order power series $\widehat{f}(t)$:

$$\widehat{f}(t) = f\left(t^n\right) = \sum_{j=0}^{+\infty} c_j t^j. \tag{2.5}$$

The set of series given by (2.1) is denoted as ${}^{C}\mathbb{F}$. Multiplication between two elements $f, g \in {}^{C}\mathbb{F}$ is defined in the Cauchy sense:

$$f \cdot g = \left(\sum_{j=0}^{+\infty} c_j w_j^{(n)}\right) \cdot \left(\sum_{j=0}^{+\infty} b_j w_j^{(n)}\right) = \sum_{j=0}^{+\infty} \left(\sum_{k=0}^{j} \binom{j/n}{k/n} c_k b_{j-k}\right) w_j^{(n)}, \tag{2.6}$$

since $w_j^{(n)} w_k^{(n)} = \binom{(j+k)/n}{k/n} w_{j+k}^{(n)}$ for any $j, k \in \mathbb{Z}_0$.

Note that the following property of the binomial coefficient is used in further sections:

$$\binom{\alpha+\beta}{\alpha} = \binom{\alpha+\beta}{\beta} = \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)}, \tag{2.7}$$

where $\alpha, \beta \geq 0$.

More details on the properties of fractional power series are given in [15, 16].

2.2. Caputo fractional derivative operator

The Caputo fractional derivative will be considered in this paper. Let $\left({}^C\boldsymbol{D}^{(1/n)}\right)^n$ denote the Caputo derivative of order $\frac{1}{n}$. The Caputo derivative acts on the basis elements (2.3):

$${}^C\boldsymbol{D}^{(1/n)} w_j^{(n)} = \begin{cases} 0, & j = 0 \\ w_{j-1}^{(n)}, & j = 1, 2, \ldots. \end{cases} \tag{2.8}$$

The Caputo derivative of order $\alpha = \frac{k}{n}$, $\gcd(k, n) = 1$ is realized via taking the kth power the operator ${}^C\boldsymbol{D}^{(1/n)}$.

2.3. The construction of analytical solutions to $\left({}^C\boldsymbol{D}^{(1/n)}\right)^n$ type FDEs

A summary of the scheme for the construction of analytical solutions to $\left({}^C\boldsymbol{D}^{(1/n)}\right)^n$ type FDEs is presented in this section. This scheme relies on the construction of an equivalent ODE via a characteristic function. The proof that a solution obtained using this scheme does satisfy the original FDE is given in [26].

Consider the following type $\left({}^C\boldsymbol{D}^{(1/n)}\right)^n$ FDE:

$$\left({}^C\boldsymbol{D}^{(1/n)}\right)^n y = F(x, y), \tag{2.9}$$

where $F(x, y)$ is bivariate analytic function. The solution to (2.9) is constructed in the form of a fractional power series (as defined in Section 2.1):

$$y = \sum_{j=0}^{+\infty} v_j w_j^{(n)} = \sum_{j=0}^{+\infty} c_j x^{\frac{j}{n}} \in {}^C\mathbb{F}. \tag{2.10}$$

Series (2.10) is convergent for $0 \leq x < R, \quad R > 0$.

Inserting (2.10) into (2.9) yields the following relation:

$$\sum_{j=0}^{+\infty} \left(1 + \frac{j}{n}\right) c_{j+n} x^{\frac{j}{n}} = F(x, y). \tag{2.11}$$

Setting $t = x^{\frac{1}{n}}$ and rearranging (2.11) results in:

$$\sum_{j=n}^{+\infty} j c_j t^{j-1} = n t^{n-1} F\left(t^n, \widehat{y}\right), \tag{2.12}$$

where $\widehat{y}$ is the integer power series that corresponds to the fractional power series (2.10):

$$\widehat{y} = \widehat{y}(t) = \sum_{j=0}^{+\infty} c_j t^j. \tag{2.13}$$

Note that (2.12) is equivalent to the following ODE:

$$\frac{\mathrm{d}\widehat{y}}{\mathrm{d}t} = nt^{n-1} F\left(t^n, \widehat{y}\right) + \sum_{j=1}^{n-1} j c_j t^{j-1}. \tag{2.14}$$

As shown in [26], inserting $t = x^{\frac{1}{n}}$ into the solution of the above equation yields a solution to the following Cauchy problem on (2.9):

$$\begin{aligned} &\left({}^{C}\boldsymbol{D}^{(1/n)}\right)^{n} y = F(x, y); \\ &y(0) = y_0; \quad \left({}^{C}\boldsymbol{D}^{(1/n)}\right)^{k} y\bigg|_{x=0} = v_k = \Gamma\left(1 + \frac{k}{n}\right) c_k, \ k = 1, \ldots, n-1. \end{aligned} \tag{2.15}$$

The initial condition of fractional derivatives at $x = 0$ is due to (2.10) and the relation:

$$\left({}^{C}\boldsymbol{D}^{(1/n)}\right)^{k} x^{\frac{k}{n}} = \Gamma\left(1 + \frac{k}{n}\right) w_0^{(n)} = \Gamma\left(1 + \frac{k}{n}\right). \tag{2.16}$$

The algorithm for solving FDE (2.9) is depicted in Figure 1. Note that [25] outlines the algorithm for numerical integration of FDE (2.9) based on the extension of fractional power series via the use of generalized differential operators.

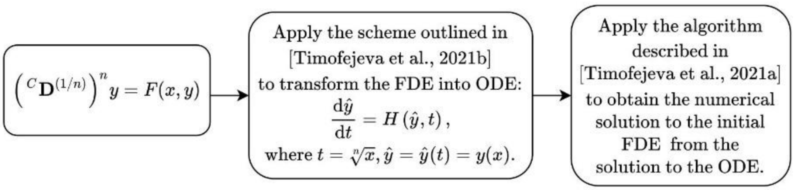


Figure 1. A schematic diagram of the algorithm in [25] to construct numerical solutions to (2.9).

Example: The Riccati equation

Consider the following Cauchy problem on the Riccati fractional differential equation with constant coefficients:

$$\begin{aligned} &\left({}^{C}\boldsymbol{D}^{(1/2)}\right)^{2} y = y^2 + y - 2; \\ &y(0) = \alpha; \quad {}^{C}\boldsymbol{D}^{(1/2)} y\bigg|_{x=0} = \beta, \end{aligned} \tag{2.17}$$

where $\alpha, \beta \in \mathbb{R}$. The solution to (2.17) is a fractional power series (2.1) with $n = 2$:

$$y = \sum_{j=0}^{+\infty} c_j \left(\sqrt{x}\right)^j. \tag{2.18}$$

The initial condition ${}^C\boldsymbol{D}^{(1/2)}y\Big|_{x=0} = \beta$ yields $c_1 = \frac{\beta}{\Gamma\left(\frac{3}{2}\right)}$. Furthermore, noting that:

$$\left({}^C\boldsymbol{D}^{(1/2)}\right)^2 y = \sum_{j=0}^{+\infty} \frac{\Gamma\left(\frac{j}{2}+2\right)}{\Gamma\left(\frac{j}{2}+1\right)} c_{j+2} \left(\sqrt{x}\right)^j = \sum_{j=0}^{+\infty} \left(1+\frac{j}{2}\right) c_{j+2} \left(\sqrt{x}\right)^j, \tag{2.19}$$

and inserting the series (2.18) into (2.17) yields:

$$\sum_{j=0}^{+\infty} \left(1+\frac{j}{2}\right) c_{j+2} \left(\sqrt{x}\right)^j = a_0 + \sum_{j=0}^{+\infty} \left(a_1 + \sum_{k=0}^{j} c_k c_{j-k}\right) \left(\sqrt{x}\right)^j. \tag{2.20}$$

Using the substitution $t = \sqrt{x}$ and denoting $\widehat{y}(t) = y\left(t^2\right)$ transforms the above equation into:

$$\sum_{j=0}^{+\infty} \left(1+\frac{j}{2}\right) c_{j+2} t^j = P\left(\widehat{y}\right), \tag{2.21}$$

where $P\left(\widehat{y}\right) = \widehat{y}^2 + \widehat{y} - 2$. Multiplying both sides of (2.21) by 2 yields:

$$\sum_{j=0}^{+\infty} (j+2)\, c_{j+2} t^j = 2P\left(\widehat{y}\right). \tag{2.22}$$

Rearranging the sum on the left-hand side of (2.22) and multiplying by t results in:

$$\sum_{j=2}^{+\infty} j c_j t^{j-1} = 2tP\left(\widehat{y}\right). \tag{2.23}$$

Finally, adding $c_1 = \frac{\beta}{\Gamma\left(\frac{3}{2}\right)}$ to both sides results in the following ODE:

$$\frac{\mathrm{d}\widehat{y}}{\mathrm{d}t} = 2t\left(\widehat{y}^2 + \widehat{y} - 2\right) + \frac{\beta}{\Gamma\left(\frac{3}{2}\right)}; \quad y(0) = \alpha. \tag{2.24}$$

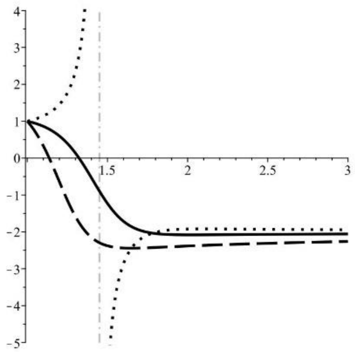
Note that the β, which is an initial condition to FDE (2.17) is a parameter in ODE (2.24).

The kink solitary solution to (2.24) is obtained for $\beta = 0$ in [29]. However, this case leads to coefficients $c_{2j+1} = 0$, $j = 0, 1, \ldots$ which in its turn results in a solution to the ODE:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 + y - 2. \tag{2.25}$$

While the kink solitary solution does indeed satisfy (2.17), the entire set of solutions to the FDE is much wider. Every solution to (2.24) for some $\beta \in \mathbb{R}$ also satisfies (2.17) after the transformation $t = \sqrt{x}$.

For $\beta \neq 0$, ODE (2.24) can only be solved in series form, via expression of solutions by confluent hypergeometric series [19]. Solutions to both (2.24) and (2.17) are depicted in parts (a) and (b) of Figure 2 respectively. Note that the scale of the x-axis changes for the FDE and ODE respectively due to the substitution $t = \sqrt{x}$. This also shifts the singularity point from its position in Figure 2 (a) to that in Figure 2 (b).



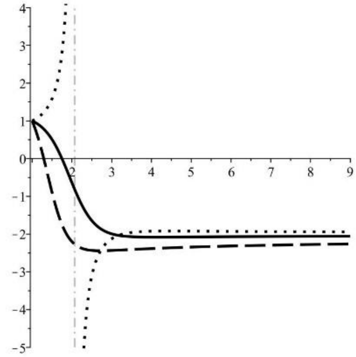


Figure 2. Solutions to (2.24) (part (a)) and (2.17) (part (b)). The initial conditions x_0, y_0 are set to 0 and 1 respectively, while $\beta = -\Gamma\left(\frac{3}{2}\right)$, $\beta = -5\Gamma\left(\frac{3}{2}\right)$, $\beta = \Gamma\left(\frac{3}{2}\right)$ for the black solid, dashed and dotted lines respectively. Note that the solutions are singular for $\beta > 0$: The grey dash-dotted line corresponds to the singularity point.

3. Main results

The main goal of the following derivations is to provide analytical techniques for the conversion of the type ${}^{C}\boldsymbol{D}^{(1/n)}$ problem into a problem of the type $\left({}^{C}\boldsymbol{D}^{(1/n)}\right)^{n}$. Without the loss of generality and for the clarity of the presentation, the denominator of the fractional derivative order will be set to $n = 2$. The presented steps can be readily generalized for different values of n.

For clarity of presentation, subsequent sections discuss the application of the described scheme on the paradigmatic example of Riccati-type FDEs. However, the analytical and numerical computations can be performed for a general FDE of type (1.2).

3.1. Auxiliary lemmas

In this section, three lemmas on the series solutions of the Riccati-type FDEs are given. The results presented here define auxiliary functions Θ_z, Φ and Ψ, which are essential for the transformation of the type ${}^{C}\boldsymbol{D}^{(1/n)}$ problem into a type $\left({}^{C}\boldsymbol{D}^{(1/n)}\right)^{\bar{n}}$ problem and, in its turn, for the construction of analytical solutions to (1.2).

Lemma 3.1. *Let $z = \sum_{j=0}^{+\infty} v_j w_j^{(2)} \in {}^{C}\mathbb{F}^{(1/2)}$ be any fractional power series. The Caputo derivative of z^2 reads:*

$$ {}^{C}\boldsymbol{D}^{(1/2)} z^2 = 2z\, {}^{C}\boldsymbol{D}^{(1/2)} z + \Theta_z(x), \tag{3.1} $$

where $\Theta_z(x) = \sum_{j=0}^{+\infty} \theta_j w_j^{(2)}$, $\theta_0 = 0$ and

$$ \theta_j = \sum_{k=1}^{j} \frac{1}{\Gamma\left(\frac{k}{2}+1\right)} \left(\frac{\Gamma\left(\frac{j+3}{2}\right)}{\Gamma\left(\frac{j-k+3}{2}\right)} - 2\frac{\Gamma\left(\frac{j}{2}+1\right)}{\Gamma\left(\frac{j-k}{2}+1\right)} \right) v_k v_{j-k+1}, \quad j = 1, 2, \ldots. \tag{3.2} $$

Proof. Inserting the fractional power series $z = \sum_{j=0}^{+\infty} v_j w_j^{(2)}$ into the left hand side of (3.1) yields:

$$ \begin{aligned} {}^{C}\boldsymbol{D}^{(1/2)} z^2 &= {}^{C}\boldsymbol{D}^{(1/2)} \left(\sum_{j=0}^{+\infty} v_j w_j^{(2)} \right)^2 = {}^{C}\boldsymbol{D}^{(1/2)} \left(\sum_{j=0}^{+\infty} \left(\sum_{k=0}^{j} \binom{j/2}{k/2} v_k v_{j-k} \right) w_j^{(2)} \right) \\ &= {}^{C}\boldsymbol{D}^{(1/2)} \left(\sum_{j=0}^{+\infty} \left(\sum_{k=0}^{j} \frac{\Gamma\left(\frac{j}{2}+1\right)}{\Gamma\left(\frac{k}{2}+1\right)\Gamma\left(\frac{j-k}{2}+1\right)} v_k v_{j-k} \right) w_j^{(2)} \right) \\ &= \sum_{j=1}^{+\infty} \left(\sum_{k=0}^{j} \frac{\Gamma\left(\frac{j}{2}+1\right)}{\Gamma\left(\frac{k}{2}+1\right)\Gamma\left(\frac{j-k}{2}+1\right)} v_k v_{j-k} \right) w_{j-1}^{(2)} \\ &= \sum_{j=0}^{+\infty} \left(\sum_{k=0}^{j+1} \frac{\Gamma\left(\frac{j+1}{2}+1\right)}{\Gamma\left(\frac{k}{2}+1\right)\Gamma\left(\frac{j-k+1}{2}+1\right)} v_k v_{j-k+1} \right) w_j^{(2)}. \end{aligned} \tag{3.3} $$

Analogously, inserting $z = \sum_{j=0}^{+\infty} v_j w_j^{(2)}$ and $\Theta_z(x) = \sum_{j=0}^{+\infty} \theta_j w_j^{(2)}$ into the right hand side of (3.1) results in:

$$
\begin{aligned}
&2z\,{}^{C}\boldsymbol{D}^{(1/2)}z + \Theta_z(x) = 2\sum_{j=0}^{+\infty} v_j w_j^{(2)} \sum_{j=1}^{+\infty} v_j w_{j-1}^{(2)} + \Theta_z(x) \\
&= 2\sum_{j=0}^{+\infty} v_j w_j^{(2)} \sum_{j=0}^{+\infty} v_{j+1} w_j^{(2)} + \Theta_z(x) = 2\sum_{j=0}^{+\infty}\left(\sum_{k=0}^{j}\binom{j/2}{k/2} v_k v_{j-k+1}\right) w_j^{(2)} + \Theta_z(x) \\
&= 2\sum_{j=0}^{+\infty}\left(\sum_{k=0}^{j}\frac{\Gamma\left(\frac{j}{2}+1\right)}{\Gamma\left(\frac{k}{2}+1\right)\Gamma\left(\frac{j-k}{2}+1\right)} v_k v_{j-k+1}\right) w_j^{(2)} \\
&+ \sum_{j=1}^{+\infty}\left(\sum_{k=1}^{j}\frac{1}{\Gamma\left(\frac{k}{2}+1\right)}\left(\frac{\Gamma\left(\frac{j+3}{2}\right)}{\Gamma\left(\frac{j-k+3}{2}\right)} - 2\frac{\Gamma\left(\frac{j}{2}+1\right)}{\Gamma\left(\frac{j-k}{2}+1\right)}\right) v_k v_{j-k+1}\right) w_j^{(2)} \\
&= 2\sum_{j=0}^{+\infty}\left(\sum_{k=0}^{j}\frac{\Gamma\left(\frac{j}{2}+1\right)}{\Gamma\left(\frac{k}{2}+1\right)\Gamma\left(\frac{j-k}{2}+1\right)} v_k v_{j-k+1}\right) w_j^{(2)} \\
&+ \sum_{j=0}^{+\infty}\left(\sum_{k=0}^{j+1}\frac{\Gamma\left(\frac{j+1}{2}+1\right)}{\Gamma\left(\frac{k}{2}+1\right)\Gamma\left(\frac{j-k+1}{2}+1\right)} v_k v_{j-k+1} - 2\sum_{k=0}^{j}\frac{\Gamma\left(\frac{j}{2}+1\right)}{\Gamma\left(\frac{k}{2}+1\right)\Gamma\left(\frac{j-k}{2}+1\right)} v_k v_{j-k+1}\right) w_j^{(2)} \\
&= \sum_{j=0}^{+\infty}\left(\sum_{k=0}^{j+1}\frac{\Gamma\left(\frac{j+1}{2}+1\right)}{\Gamma\left(\frac{k}{2}+1\right)\Gamma\left(\frac{j-k+1}{2}+1\right)} v_k v_{j-k+1}\right) w_j^{(2)}.
\end{aligned}
\tag{3.4}
$$

□

Note that the function $\Theta_z(x) = {}^{C}\boldsymbol{D}^{(1/2)}z^2 - 2z\,{}^{C}\boldsymbol{D}^{(1/2)}z$ quantifies the effect of fractional differentiation of z^2. If ${}^{C}\boldsymbol{D}^{(1/2)}$ is replaced by an integer order derivative $\frac{\mathrm{d}}{\mathrm{d}x}$, the function $\Theta_z(x)$ becomes equal to zero.

The two following lemmas yield results on coefficients of the fractional power series solutions of the Riccati-type problems. Note that while the solution coefficients can be directly computed using these results, the evaluation of the solution does not readily follow (different numerical algorithms, such as described in [25] could be used for the evaluation).

Lemma 3.2. *Consider the following Cauchy problem with respect to the Riccati fractional differential equation:*

$$
\begin{aligned}
{}^{C}\boldsymbol{D}^{(1/2)}y_1 &= a_2 y_1^2 + a_1 y_1 + a_0 + \Phi(x); \\
y_1(0) &= \gamma_0,
\end{aligned}
\tag{3.5}
$$

where $a_2, a_1, a_0, \gamma_0 \in \mathbb{R}$, and $\Phi(x)$ is a given fractional power series with coefficients $\phi_j \in \mathbb{R}$:

$$
\Phi(x) = \sum_{j=1}^{+\infty} \phi_j w_j^{(2)} \in {}^{C}\mathbb{F}^{(1/2)}. \tag{3.6}
$$

The solution to (3.5) reads:

$$
y_1 = \sum_{j=0}^{+\infty} \gamma_j w_j^{(2)}, \quad \gamma_j \in \mathbb{R}, \tag{3.7}
$$

where,

$$\gamma_1 = a_2\gamma_0^2 + a_1\gamma_0 + a_0,$$
$$\gamma_{j+1} = a_2\left(\sum_{\substack{k_1,k_2=0,1,\ldots\\k_1+k_2=j}} \frac{\Gamma\left(\frac{j}{2}+1\right)\gamma_{k_1}\gamma_{k_2}}{\Gamma\left(\frac{k_1}{2}+1\right)\Gamma\left(\frac{k_2}{2}+1\right)}\right) + a_1\gamma_j + \phi_j. \quad j = 1,2,\ldots. \tag{3.8}$$

Proof. Coefficients (3.8) are obtained by inserting the fractional power series (3.7) into (3.5). □

Lemma 3.3. *Consider the following Cauchy problem:*

$$\left({}^C\boldsymbol{D}^{(1/2)}\right)^2 y_2 = b_3y_2^3 + b_2y_2^2 + b_1y_2 + \Psi(x);$$
$$y_2(0) = \lambda_0; \quad {}^C\boldsymbol{D}^{(1/2)}y_2\Big|_{x=0} = \lambda_1, \tag{3.9}$$

where $b_3, b_2, b_1, \lambda_0, \lambda_1 \in \mathbb{R}$ and $\Psi(x)$ is a given fractional power series with coefficients $\psi_j \in \mathbb{R}$:

$$\Psi(x) = \sum_{j=0}^{+\infty} \psi_j w_j^{(2)} \in {}^C\mathbb{F}^{(1/2)}. \tag{3.10}$$

The solution to (3.9) reads:

$$y_2 = \sum_{j=0}^{+\infty} \lambda_j w_j^{(2)}, \quad \lambda_j \in \mathbb{R}, \tag{3.11}$$

where,

$$\lambda_{j+2} = b_3\left(\sum_{\substack{k_1,k_2,k_3=0,1,\ldots\\k_1+k_2+k_3=j}} \frac{\Gamma\left(\frac{j}{2}+1\right)\lambda_{k_1}\lambda_{k_2}\lambda_{k_3}}{\Gamma\left(\frac{k_1}{2}+1\right)\Gamma\left(\frac{k_2}{2}+1\right)\Gamma\left(\frac{k_3}{2}+1\right)}\right)$$
$$+ b_2\left(\sum_{\substack{k_1,k_2=0,1,\ldots\\k_1+k_2=j}} \frac{\Gamma\left(\frac{j}{2}+1\right)\lambda_{k_1}\lambda_{k_2}}{\Gamma\left(\frac{k_1}{2}+1\right)\Gamma\left(\frac{k_2}{2}+1\right)}\right) + b_1\lambda_j + \psi_j, \quad j = 0,1,\ldots. \tag{3.12}$$

Proof. Coefficients (3.12) are obtained by inserting the fractional power series (3.11) into (3.9). □

3.2. The construction of solutions to the fractional Riccati equation

The results obtained in section 3.1 are now applied to derive the relationship between problems (3.5) and (3.9) as well as their respective solutions.

Consider the fractional Riccati equation (3.5). Differentiating (3.5) via the operator ${}^C\boldsymbol{D}^{(1/2)}$ yields:

$$\left({}^C\boldsymbol{D}^{(1/2)}\right)^2 y_1 = a_2\,{}^C\boldsymbol{D}^{(1/2)}y_1^2 + a_1\,{}^C\boldsymbol{D}^{(1/2)}y_1 + {}^C\boldsymbol{D}^{(1/2)}\Phi(x). \tag{3.13}$$

Applying Lemma 3.1 to the first term of the right hand side of (3.13) yields:

$$\begin{aligned}\left({}^{C}\boldsymbol{D}^{(1/2)}\right)^{2} y_1 &= 2a_2 y_1\,{}^{C}\boldsymbol{D}^{(1/2)} y_1 + a_2\Theta_{y_1}(x) + a_1\,{}^{C}\boldsymbol{D}^{(1/2)} y_1 + {}^{C}\boldsymbol{D}^{(1/2)}\Phi(x)\\ &= (2a_2 y_1 + a_1)\,{}^{C}\boldsymbol{D}^{(1/2)} y_1 + {}^{C}\boldsymbol{D}^{(1/2)}\Phi(x) + a_2\Theta_{y_1}(x).\end{aligned} \tag{3.14}$$

Inserting ${}^{C}\boldsymbol{D}^{(1/2)} y_1 = a_2 y_1^2 + a_1 y_1 + a_0 + \Phi(x)$ transforms (3.14) into:

$$\begin{aligned}\left({}^{C}\boldsymbol{D}^{(1/2)}\right)^{2} y_1 =& 2a_2^2 y_1^3 + 3a_1 a_2 y_1^2 + \left(a_1^2 + 2a_0 a_2\right) y_1\\ &+ {}^{C}\boldsymbol{D}^{(1/2)}\Phi(x) + a_2\Theta_{y_1}(x) + 2a_2 y_1\Phi(x) + a_1\left(a_0 + \Phi(x)\right).\end{aligned} \tag{3.15}$$

Let us consider the following notation:

$$\Psi(x) = {}^{C}\boldsymbol{D}^{(1/2)}\Phi(x) + a_2\Theta_{y_1}(x) + 2a_2 y_1\Phi(x) + a_1\left(a_0 + \Phi(x)\right). \tag{3.16}$$

The function $\Psi(x)$ is utilized in constructing solutions to FDE (3.5), while functions $\Phi(x)$ and $\Theta_{y_1}(x)$ are given in (3.5) and obtained from (3.1) respectively. Note that $\Psi(x)$ simplifies to a linear function of $\Theta_{y_1}(x)$ if $\Phi(x) = 0$.

Comparing (3.15) and (3.9) yields the following relationship between coefficients a_0, a_1, a_2 and $b_0, \ldots, b_3$:

$$\begin{aligned} b_3 &= 2a_2^2;\\ b_2 &= 3a_1 a_2;\\ b_1 &= a_1^2 + 2a_0 a_2.\end{aligned} \tag{3.17}$$

Applying (3.16) and (3.17) transforms (3.15) into:

$$\left({}^{C}\boldsymbol{D}^{(1/2)}\right)^{2} y_1 = b_3 y_1^3 + b_2 y_1^2 + b_1 y_1 + \Psi(x). \tag{3.18}$$

Note, that (3.18) has the same form as (3.9).

Moreover, (3.16) induces the following relations between the coefficients ψ_j, ϕ_j, θ_j and γ_j, $j = 0, 1, \ldots$:

$$\begin{aligned} \psi_0 &= \phi_1 + a_0 a_1;\\ \psi_j &= \phi_{j+1} + a_2\theta_j + a_1\phi_j + 2a_2\sum_{k=0}^{j}\binom{j/2}{k/2}\gamma_k\phi_{j-k}; \quad j = 1, 2, \ldots.\end{aligned} \tag{3.19}$$

The above derivations result in the following theorem.

Theorem 3.1. *Cauchy problems (3.5) and (3.9) have the same solution* $y_1 = y_2 = y = \sum_{j=0}^{+\infty}\gamma_j w_j^{(2)}$ *if relations (3.16), (3.17) and the following equalities:*

$$\lambda_0 = \gamma_0; \quad \lambda_1 = a_2\gamma_0^2 + a_1\gamma_0 + a_0 \tag{3.20}$$

do hold true.

Having derived the relationship between these problems, existing algorithms can be applied to solve (3.9), as detailed in Figure 3.

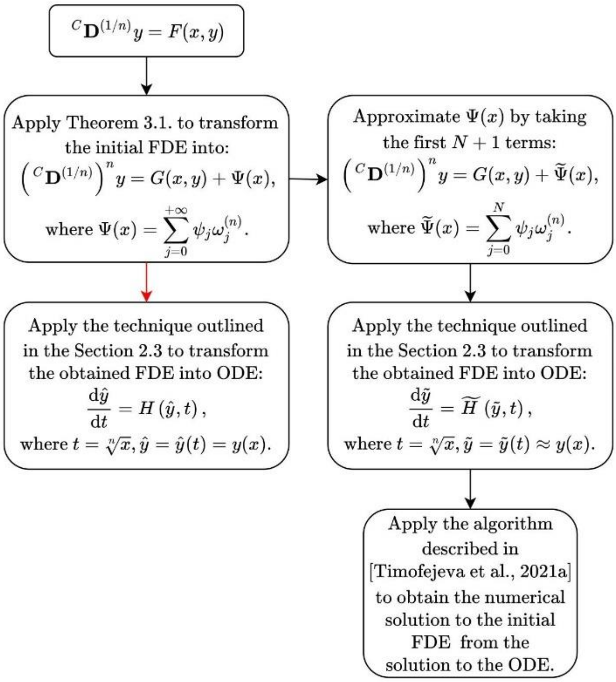


Figure 3. The schematic diagram of the algorithm for transforming ${}^{C}\boldsymbol{D}^{(1/n)}$-type FDEs into $\left({}^{C}\boldsymbol{D}^{(1/n)}\right)^{n}$-type FDEs. The red line depicts an algorithm step that cannot be practically implemented, as $\Psi(x)$ is an infinite series, prompting the requirement to truncate $\Psi(x)$.

4. Computational experiments

Consider the following Cauchy problem with respect to the Riccati fractional differential equation:

$$
\begin{aligned}
{}^{C}\boldsymbol{D}^{(1/2)}y &= \frac{1}{4}y^2 + \frac{1}{2}y - \frac{1}{3}; \\
y(0) &= \frac{1}{10}.
\end{aligned}
\tag{4.1}
$$

Note that in this case, the function $\Phi(x) = 0$, thus, $\phi_j = 0$; $j = 0, 1, \ldots$.

Using the Theorem 3.1, the values of the parameters b_1, b_2, b_3 and the initial conditions λ_0, λ_1 can be computed as follows:

$$\begin{aligned} b_3 &= 2\cdot\left(\frac{1}{4}\right)^2 = \frac{1}{8};\\ b_2 &= 3\cdot\frac{1}{2}\cdot\frac{1}{4} = \frac{3}{8};\\ b_1 &= \left(\frac{1}{2}\right)^2 + 2\cdot\left(-\frac{1}{3}\right)\cdot\frac{1}{4} = \frac{1}{12};\\ \lambda_0 &= \frac{1}{10};\\ \lambda_1 &= \frac{1}{4}\cdot\left(\frac{1}{10}\right)^2 + \frac{1}{2}\cdot\frac{1}{10} - \frac{1}{3} = -\frac{337}{1200}. \end{aligned} \tag{4.2}$$

Thus, (4.1) can be transformed into the following Cauchy problem:

$$\begin{aligned} &\left({}^C\boldsymbol{D}^{(1/2)}\right)^2 y = \frac{1}{8}y^3 + \frac{3}{8}y^2 + \frac{1}{12}y + \Psi(x);\\ &y(0) = \frac{1}{10};\quad {}^C\boldsymbol{D}^{(1/2)}y\Big|_{x=0} = -\frac{337}{1200}, \end{aligned} \tag{4.3}$$

where the coefficients of the function $\Psi(x) = \sum_{j=0}^{+\infty}\psi_j w_j^{(2)}$ are obtained using relations (3.19).

Following the technique outlined in Section 2.3, (4.3) can be converted into the following ODE:

$$\frac{\mathrm{d}\widehat{y}}{\mathrm{d}t} = 2t\left(\frac{1}{8}\widehat{y}^3 + \frac{3}{8}\widehat{y}^2 + \frac{1}{12}\widehat{y} + \Psi(t^2)\right) - \frac{337}{1200\Gamma(\frac{3}{2})};\quad \widehat{y}(0) = \frac{1}{10}, \tag{4.4}$$

where $t = \sqrt{x}$ and $\widehat{y} = \widehat{y}(t) = y(x)$. Note that the function $\Psi(x)$ is changed into $\Psi\left(t^2\right)$ due to the independent variable substitution. The function $\Psi(t^2)$ can only be represented by an infinite power series (a known closed form of $\Psi\left(t^2\right)$ does not exist). Thus, the above ODE cannot be solved directly. To integrate (4.4), $\Psi(t^2)$ is approximated taking the first $N+1$ terms:

$$\frac{\mathrm{d}\widetilde{y}}{\mathrm{d}t} = 2t\left(\frac{1}{8}\widetilde{y}^3 + \frac{3}{8}\widetilde{y}^2 + \frac{1}{12}\widetilde{y} + \sum_{j=0}^{N}\psi_j\frac{t^{2j}}{\Gamma\left(1+\frac{j}{n}\right)}\right) - \frac{337}{1200\Gamma(\frac{3}{2})};\quad \widetilde{y}(0) = \frac{1}{10}, \tag{4.5}$$

where $\widetilde{y}$ tends to $\widehat{y}$ as N tends to infinity.

It is clear that the approximation of the series Ψ via the polynomial $\sum_{j=0}^{N}\psi_j\frac{t^{2j}}{\Gamma\left(1+\frac{j}{n}\right)}$ introduces errors into the solution. Exact expressions and approximate numerical values of the coefficients $\psi_j\,(j = 0, 1, \ldots, 8)$ are given in the Appendix A.

The solution to FDE (4.1) can now be obtained from the solution to the ODE (4.5) via the algorithm described in [25]. Figure 4 (a,b) depicts the solutions to (4.5) and (4.1) respectively for different values of N. These solutions are compared with a direct numerical solution computed via Garrappa's method [8] to (4.1) in Figure 4 (b). It can be seen from Figure 4 that increasing N does cause the solution to converge, although that convergence is not monotonous.

In general, any numerical method can be used to construct solutions to (4.5). However, using the semi-analytical scheme presented in [25] makes it easier to perform the transformation of the time-axis, since the solution to (4.5) is given as a piecewise-polynomial function. If a purely numerical method is used (such as the classical Euler method, or any Runge-Kutta class technique), the nonlinear time axis transformation needs to be taken into consideration when selecting the numerical integration step-size.

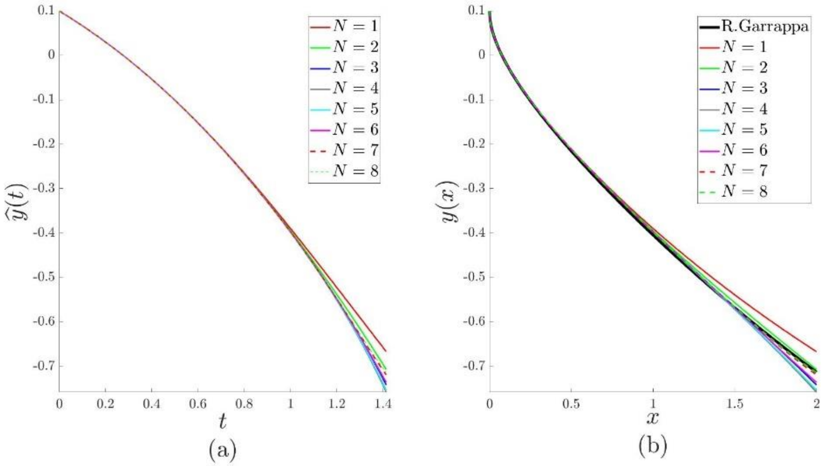


Figure 4. Convergence of the numerical solution to (4.1). Part (a) depicts the approximate solutions to the ODE (4.4) for various values of $N = 1, 2, \ldots, 8$, while part (b) depicts the approximate solutions to the FDE (4.1) for $N = 1, 2, \ldots, 8$. The obtained solutions are compared to a direct numerical integrator result [8] (black solid line).

Consider the following Cauchy problem with initial condition being equal to γ_0:

$$\begin{aligned} {}^{C}\boldsymbol{D}^{(1/2)}y &= \frac{1}{4}y^2 + \frac{1}{2}y - \frac{1}{3}; \\ y(0) &= \gamma_0. \end{aligned} \tag{4.6}$$

The root mean square error (RMSE) between solutions computed via the presented algorithm (denoted $y(x)$) and Garrappa's method (denoted $y_G(x)$) is defined as:

$$\mathrm{RMSE}\,(y, y_G) = \sqrt{\frac{1}{M+1}\sum_{j=0}^{M}\left(y(jh) - y_G(jh)\right)^2}, \tag{4.7}$$

where h denotes the integration step size; M is the number of integration steps.

It can be seen in Figure 5 that for the initial condition $\gamma_0 \in [0.1, 0.3]$, RMSE between Garrappa's solution and solutions obtained by truncating $\Psi(x)$ at $N = 1, \ldots, 8$ significantly decreases up to $N = 4$.

Using a higher-order approximation for $\Psi(x)$ than $N = 4$ does not yield a significant improvement RMSE-wise.

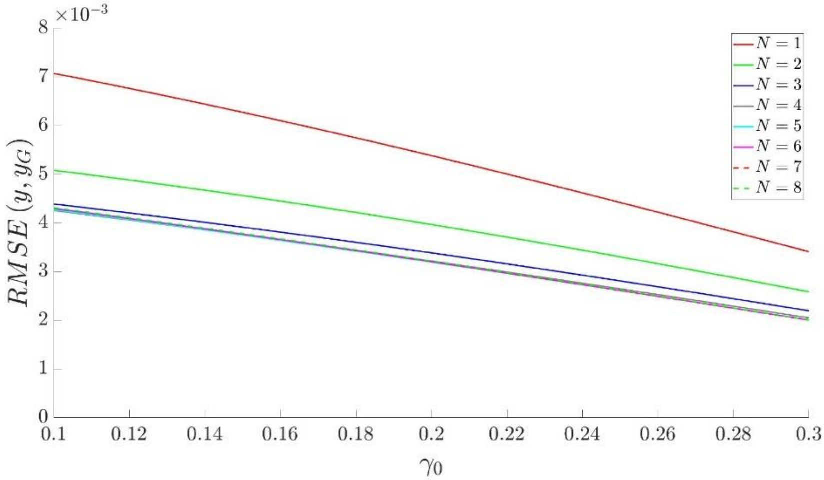


Figure 5. The root mean square error between Garrappa's solution to (4.1) and solution obtained by truncating $\Psi(x)$ at $N = 1, \ldots, 8$.

5. Conclusions

This paper proposes a new approach for solving ${}^{C}\boldsymbol{D}^{(1/n)}$-type FDEs. The construction of analytical solutions to a general form FDE without a direct evaluation of Caputo type integrals is a demanding mathematical problem. It has been demonstrated that some $\left({}^{C}\boldsymbol{D}^{(1/n)}\right)^{n}$-type FDEs can be solved by transforming them into ODEs and applying a numerical algorithm [26].

The main contribution of this paper is the extension of the class of FDEs where similar fractional power series can be applied: The scheme is no longer limited to $\left({}^{C}\boldsymbol{D}^{(1/n)}\right)^{n}$-type FDEs, but can be applied to ${}^{C}\boldsymbol{D}^{(1/n)}$-type FDEs. It opens new possibilities for the generation of solutions to such FDEs which previously could be analyzed using only approximate numerical techniques. Difficulties related to the application of the proposed technique are discussed in the paper and the presented numerical examples demonstrate the efficacy of the proposed technique.

Conflict of interest

The authors declare that they have no competing interests.

References

1. R. Alchikh, S. Khuri, Numerical solution of a fractional differential equation arising in optics, *Optik*, **208** (2020), 163911. https://doi.org/10.1016/j.ijleo.2019.163911
2. M. I. Asjad, W. A. Faridi, K. M. Abualnaja, A. Jhangeer, H. Abu-Zinadah, H. Ahmad, The fractional comparative study of the non-linear directional couplers in non-linear optics, *Results Phys.*, **27** (2021), 104459. https://doi.org/10.1016/j.rinp.2021.104459
3. M. S. Asl, M. Javidi, Novel algorithms to estimate nonlinear FDEs: Applied to fractional order nutrient-phytoplankton-zooplankton system, *J. Comput. Appl. Math.*, **339** (2018), 193–207. https://doi.org/10.1016/j.cam.2017.10.030
4. D. Baleanu, H. Mohammadi, S. Rezapour, A fractional differential equation model for the COVID-19 transmission by using the Caputo-Fabrizio derivative, *Adv. Differ. Equ.*, **2020** (2020), 1–27. https://doi.org/10.1186/s13662-020-02762-2
5. A. De Gaetano, S. Sakulrang, A. Borri, D. Pitocco, S. Sungnul, E. J. Moore, Modeling continuous glucose monitoring with fractional differential equations subject to shocks, *J. Theor. Biol.*, **526** (2021), 110776. https://doi.org/10.1016/j.jtbi.2021.110776
6. N. J. Ford, A. C. Simpson, The numerical solution of fractional differential equations: Speed versus accuracy, *Numer. Algorithms*, **26** (2001), 333–346. https://doi.org/10.1023/A:1016601312158
7. L. Frunzo, R. Garra, A. Giusti, V. Luongo, Modeling biological systems with an improved fractional gompertz law, *Commun. Nonlinear Sci. Numer. Simul.*, **74** (2019), 260–267. https://doi.org/10.1016/j.cnsns.2019.03.024
8. R. Garrappa, Numerical solution of fractional differential equations: A survey and a software tutorial, *Mathematics*, **6** (2018), 16. https://doi.org/10.3390/math6020016
9. R. Hilfer, *Applications of fractional calculus in physics*, World Scientific, 2000. https://doi.org/10.1142/3779
10. G. Ismail, H. R. Abdl-Rahim, A. Abdel-Aty, R. Kharabsheh, W. Alharbi, M. Abdel-Aty, An analytical solution for fractional oscillator in a resisting medium, *Chaos, Solitons Fract.*, **130** (2020), 109395. https://doi.org/10.1016/j.chaos.2019.109395
11. M. D. Johansyah, A. K. Supriatna, E. Rusyaman, J. Saputra, Application of fractional differential equation in economic growth model: A systematic review approach, *AIMS Math.*, **6** (2021), 10266–10280. https://doi.org/10.3934/math.2021594
12. R. C. Koeller, Applications of fractional calculus to the theory of viscoelasticity, *J. Appl. Mech.*, **51** (1984), 299–307. https://doi.org/10.1115/1.3167616
13. D. Kumar, J. Singh, *Fractional calculus in medical and health science*, CRC Press, 2020. https://doi.org/10.1201/9780429340567
14. C. Li, M. Cai, *Theory and numerical approximations of fractional integrals and derivatives*, SIAM, 2019. https://doi.org/10.1137/1.9781611975888
15. Z. Navickas, T. Telksnys, R. Marcinkevicius, M. Ragulskis, Operator-based approach for the construction of analytical soliton solutions to nonlinear fractional-order differential equations, *Chaos, Solitons Fract.*, **104** (2017), 625–634. https://doi.org/10.1016/j.chaos.2017.09.026

16. Z. Navickas, T. Telksnys, I. Timofejeva, R. Marcinkevičius, M. Ragulskis, An operator-based approach for the construction of closed-form solutions to fractional differential equations, *Math. Model. Anal.*, **23** (2018), 665–685. https://doi.org/10.3846/mma.2018.040

17. J. J. Nieto, Solution of a fractional logistic ordinary differential equation, *Appl. Math. Lett.*, **123** (2022), 107568. https://doi.org/10.1016/j.aml.2021.107568

18. F. Norouzi, G. M. N'Guérékata, A study of ψ-Hilfer fractional differential system with application in financial crisis, *Chaos, Solitons Fract.: X*, **6** (2021), 100056. https://doi.org/10.1016/j.csfx.2021.100056

19. F. W. Olver, D. W. Lozier, R. Boisvert, C. W. Clark, *NIST handbook of mathematical functions*, Cambridge university press, 2010.

20. R. Shah, H. Khan, M. Arif, P. Kumam, Application of Laplace-Adomian decomposition method for the analytical solution of third-order dispersive fractional partial differential equations, *Entropy*, **21** (2019), 335. https://doi.org/10.3390/e21040335

21. R. Shah, H. Khan, D. Baleanu, P. Kumam, M. Arif, A novel method for the analytical solution of fractional Zakharov-Kuznetsov equations, *Adv. Differ. Equ.*, **2019** (2019), 1–14. https://doi.org/10.1186/s13662-019-2441-5

22. R. Shanker Dubey, P. Goswami, Analytical solution of the nonlinear diffusion equation, *Eur. Phys. J. Plus*, **133** (2018), 1–12. https://doi.org/10.1140/epjp/i2018-12010-6

23. J. Singh, A. Gupta, D. Baleanu, On the analysis of an analytical approach for fractional Caudrey-Dodd-Gibbon equations, *Ale. Eng. J.*, **61** (2022), 5073–5082. https://doi.org/10.1016/j.aej.2021.09.053

24. S. Thirumalai, R. Seshadri, S. Yuzbasi, Spectral solutions of fractional differential equations modelling combined drug therapy for hiv infection, *Chaos, Solitons Fract.*, **151** (2021), 111234. https://doi.org/10.1016/j.chaos.2021.111234

25. I. Timofejeva, Z. Navickas, T. Telksnys, R. Marcinkevicius, M. Ragulskis, An operator-based scheme for the numerical integration of FDEs, *Mathematics*, **9** (2021), 1372. https://doi.org/10.3390/math9121372

26. I. Timofejeva, Z. Navickas, T. Telksnys, R. Marcinkevičius, X. J. Yang, M. Ragulskis, The extension of analytic solutions to FDEs to the negative half-line, *AIMS Math.*, **6** (2021), 3257–3271. https://doi.org/10.3934/math.2021195

27. Y. Wei, X. Li, Y. He, Generalisation of tea moisture content models based on vnir spectra subjected to fractional differential treatment, *Biosyst. Eng.*, **205** (2021), 174–186. https://doi.org/10.1016/j.biosystemseng.2021.03.006

28. C. Wen, J. Yang, Complexity evolution of chaotic financial systems based on fractional calculus, *Chaos, Solitons Fract.*, **128** (2019), 242–251. https://doi.org/10.1016/j.chaos.2019.08.005

29. V. F. Zaitsev, A. D. Polyanin, *Handbook of exact solutions for ordinary differential equations*, CRC Press, 2002.

30. J. Zhang, X. Fu, H. Morris, Construction of indicator system of regional economic system impact factors based on fractional differential equations, *Chaos, Solitons Fract.*, **128** (2019), 25–33. https://doi.org/10.1016/j.chaos.2019.07.036

31. Q. Zhou, A. Sonmezoglu, M. Ekici, M. Mirzazadeh, Optical solitons of some fractional differential equations in nonlinear optics, *J. Mod. Optics*, **64** (2017), 2345–2349. https://doi.org/10.1080/09500340.2017.1357856

Appendix A

Exact expressions of the coefficients ψ_j $(j = 0, 1, \ldots, 8)$ in (4.5) are as follows:

$$\psi_0 = -\frac{1}{6};$$

$$\psi_1 = -\frac{113569\pi - 227138}{2880000\pi};$$

$$\psi_2 = \frac{1249259\pi - 9994072}{115200000\pi};$$

$$\psi_3 = -\frac{742059846\pi^2 - 1893308799\pi + 1224728096}{20736000000\pi^2};$$

$$\psi_4 = \frac{33336476415\pi^2 - 358272490092\pi + 431104289792}{6635520000000\pi^2};$$

$$\psi_5 = -\frac{4410927401265\pi^3 - 35936061966618\pi^2 + 72641225920964\pi}{497664000000000\pi^3} + \frac{34669602941568}{497664000000000\pi^3};$$

$$\psi_6 = -\frac{1302276783665715\pi^3 - 2329090441231004\pi^2 - 41702172928558208\pi}{42467328000000000\pi^3} - \frac{32543200627818496}{42467328000000000\pi^3};$$

$$\psi_7 = \frac{4473972710859338032 5\pi^4 + 81636161754007129500\pi^3 - 762603888431624570496\pi^2}{401316249600000000000\pi^4} + \frac{1151771271228481928448\pi - 446658387089593204736}{401316249600000000000\pi^4};$$

$$\psi_8 = -\frac{15444787735972141568752 5\pi^4 - 137855111121483360954426 0\pi^3}{41094783959040000000000000\pi^4} + \frac{4294480908591536446544 64\pi^2 - 8580504348725618463424512\pi}{41094783959040000000000000\pi^4} + \frac{5031160072177177858146304}{41094783959040000000000000\pi^4}.$$

120

Table A. Values of coefficients ψ_j approximated to a precision of 10^{-6} obtained via (3.19).

j	ψ_j
0	−0.166667
1	−0.014329
2	−0.016770
3	−0.012707
4	−0.005580
5	0.001579
6	0.006157
7	0.006483
8	0.002501

CURRICULUM VITAE

Inga Telksnienė (née Timofejeva)
inga.telksniene@ktu.lt

Education:

2014 – 2018 Bachelor's Degree in Applied Mathematics, KTU
2018 – 2020 Master's Degree in Applied Mathematics, KTU
2020 – currently PhD studies in Informatics, KTU

Work experience:

2016 – 2019 Mathematics Tutor, Career and Education Centre, KTU
2018 – 2020 Laboratory Assistant, MGMF, KTU
2020 – today Junior Researcher, Center for Nonlinear Systems, KTU
2020 – today Mathematics Lecturer, Department of Mathematical Modelling, KTU

Scientific interest fields:

Nonlinear dynamics, differential equations, computer algebra

Scientific publications related to the thesis:

1. Navickas, Zenonas, et al. The Fractal Structure of Analytical Solutions to Fractional Riccati Equation. Fractals, doi: 10.1142/S0218348X23401308

2. Timofejeva, Inga, et al. The Extension of Analytic Solutions to FDEs to the Negative Half-Line. *AIMS Mathematics 6.4* (2021): 3257-3271.
3. Timofejeva, Inga, et al. An Operator-Based Scheme for the Numerical Integration of FDEs. *Mathematics 9.12* (2021): 1372.
4. Marcinkevicius, Romas, et al. The Construction of Solutions to ${}^{C}\mathbf{D}^{(1/n)}$ Type FDEs via Reduction to $\left({}^{C}\mathbf{D}^{\left(\frac{1}{n}\right)}\right)^{n}$ type FDEs. AIMS Mathematics 7.9 (2022): 16536-16554, doi: 10.3934/math.2022905.
5. Timofejeva, Inga, et al. Finite-Time Stabilization of the Fractional Model of the Driven Dissipative Nonlinear Pendulum. *International Journal of Bifurcation and Chaos 32.04* (2022): 2250056.

Participation in Scientific Conferences:

1. 20th International Conference of Numerical Analysis and Applied Mathematics (ICNAAM 2022)
2. 26th International Conference Mathematical Modelling and Analysis (MMA2023)

AUTHOR'S LIST OF PUBLICATIONS

1. **Timofejeva, I.**; Telksnys, T.; Navickas, Z.; Marcinkevičius, R.; Mickevicius, R.; Ragulskis, M. Solitary solutions to a metastasis model represented by two systems of coupled Riccati equations // Journal of King Saud University – Science. Amsterdam: Elsevier. ISSN 1018-3647. eISSN 2213-686X. 2023, vol. 35, iss. 5, art. No. 102682, p. 730–745. doi: 10.1016/j.jksus.2023.102682. [Science Citation Index Expanded (Web of Science); Scopus] [IF: 3.800; AIF: 7.200; IF/AIF: 0.527; Q2 (2022, InCites JCR SCIE)] [Field: N 001; N 009; M 001] [Input: 0.170]
2. Uzdila, Ernestas; **Telksniene, Inga**; Telksnys, Tadas; Ragulskis, Minvydas. Finite-time stabilization of unstable orbits in the fractional difference logistic map // Fractal and fractional. Basel: MDPI. ISSN 2504-3110. 2023, vol. 7, iss. 8, art. No. 570, p. 1–13. doi: 10.3390/fractalfract7080570. [Science Citation Index Expanded (Web of Science); Scopus] [IF: 5.400; AIF: 3.400; IF/AIF: 1.588; Q1 (2022, InCites JCR SCIE)] [Field: N 001] [Input: 0.250]
3. Marcinkevicius, R.; **Telksniene, I.**; Telksnys, T.; Navickas, Z.; Ragulskis, M. The construction of solutions to CD(1/n) type FDEs via reduction to (CD(1/n))n type FDEs // AIMS Mathematics. Springfield, MO: AIMS press. ISSN 2473-6988. 2022, vol. 7, iss. 9, p. 16536–16554. doi: 10.3934/math.2022905. [Science Citation Index Expanded (Web of Science); Scopus; Dimensions] [IF: 2.200; AIF: 1.650; IF/AIF: 1.333; Q1 (2022, InCites JCR SCIE)] [Field: N 001; N 009] [Input: 0.200]
4. Navickas, Zenonas; **Telksnienė, Inga**; Telksnys, Tadas; Marcinkevicius, Romas; Ragulskis, Minvydas. Construction of special soliton solutions to the stochastic Riccati equation // Open mathematics. Berlin: Walter de Gruyter. ISSN 2391-5455. 2022, vol. 20, iss. 1, p. 829–844. doi: 10.1515/math-2022-0051. [Science Citation Index Expanded (Web of Science); Scopus; Dimensions] [IF: 1.700; AIF: 1.300; IF/AIF: 1.307; Q1 (2022, InCites JCR SCIE)] [Field: N 001; T 007] [Input: 0.200]
5. Palevičius, Paulius; Pal, Mayur; Landauskas, Mantas; Orinaitė, Ugnė; **Timofejeva, Inga**; Ragulskis, Minvydas. Automatic detection of cracks on concrete surfaces in the presence of shadows // Sensors. Basel: MDPI. ISSN 1424-8220. 2022, vol. 22, iss. 10, art. No. 3662, p. 1–13. doi: 10.3390/s22103662. [Science Citation Index Expanded (Web of Science); Scopus; MEDLINE] [IF: 3.900; AIF: 4.333; IF/AIF: 0.900; Q2 (2022, InCites JCR SCIE)] [Field: N 001] [Input: 0.166]
6. **Timofejeva, Inga**; Laukaitis, Giedrius; Rinkevicius, Zilvinas; Ragulskis, Minvydas. Finite-time stabilization of the fractional model of the driven

dissipative nonlinear pendulum // International journal of bifurcation and chaos. Singapore: World scientific. ISSN 0218-1274. eISSN 1793-6551. 2022, vol. 32, iss. 4, art. No. 2250056, p. 1–20. doi: 10.1142/S0218127422500560. [Science Citation Index Expanded (Web of Science); Scopus; Academic Search Ultimate] [IF: 2.200; AIF: 5.300; IF/AIF: 0.415; Q2 (2022, InCites JCR SCIE)] [Field: N 001; N 002] [Input: 0.250]

7. Pal, Mayur; Palevičius, Paulius; Landauskas, Mantas; Orinaitė, Ugnė; **Timofejeva, Inga**; Ragulskis, Minvydas. An overview of challenges associated with automatic detection of concrete cracks in the presence of shadows // Applied sciences. Basel: MDPI. ISSN 2076-3417. 2021, vol. 11, iss. 23, art. No. 11396, p. 1–13. doi: 10.3390/app112311396. [Science Citation Index Expanded (Web of Science); Scopus; DOAJ] [IF: 2.838; AIF: 5.795; IF/AIF: 0.489; Q2 (2021, InCites JCR SCIE)] [Field: N 001; N 009] [Input: 0.166]
8. Sakalauskas, Eligijus; **Timofejeva, Inga**; Kilčiauskas, Aušrys. Sigma identification protocol construction based on MPF // Symmetry. Basel: MDPI. ISSN 2073-8994. 2021, vol. 13, iss. 9, art. No. 1683, p. 1–16. doi: 10.3390/sym13091683. [Science Citation Index Expanded (Web of Science); Scopus; DOAJ] [IF: 2.940; AIF: 7.694; IF/AIF: 0.382; Q2 (2021, InCites JCR SCIE)] [Field: N 001; N 009] [Input: 0.300]
9. **Timofejeva, Inga**; Navickas, Zenonas; Telksnys, Tadas; Marcinkevicius, Romas; Ragulskis, Minvydas. An operator-based scheme for the numerical integration of FDEs // Mathematics. Basel: MDPI. ISSN 2227-7390. 2021, vol. 9, iss. 12, art. No. 1372, p. 1–17. doi: 10.3390/math9121372. [Science Citation Index Expanded (Web of Science); Scopus; DOAJ] [IF: 2.592; AIF: 1.291; IF/AIF: 2.007; Q1 (2021, InCites JCR SCIE)] [Field: N 001; N 009] [Input: 0.200]
10. **Timofejeva, Inga**; McCraty, Rollin; Atkinson, Mike; Alabdulgader, Abdullah A.; Vainoras, Alfonsas; Landauskas, Mantas; Siauciunaite, Vaiva; Ragulskis, Minvydas. Global study of human heart rhythm synchronization with the Earth's time varying magnetic field // Applied sciences. Basel: MDPI. ISSN 2076-3417. 2021, vol. 11, iss. 7, art. No. 2935, p. 1–29. doi: 10.3390/app11072935. [Science Citation Index Expanded (Web of Science); Scopus; DOAJ] [IF: 2.838; AIF: 5.795; IF/AIF: 0.489; Q2 (2021, InCites JCR SCIE)] [Field: N 001; N 011] [Input: 0.125]
11. **Timofejeva, Inga**; Telksnys, Tadas; Navickas, Zenonas; Marcinkevicius, Romas; Ragulskis, Minvydas. Higher order solitary solutions to the meta-model of diffusively coupled Lotka-Volterra systems // Advances in difference equations. London: Springer Open. ISSN 1687-1847. 2021, vol.

2021, iss. 1, art. No. 133, p. 1–19. doi: 10.1186/s13662-021-03300-4. [Science Citation Index Expanded (Web of Science); Scopus; MEDLINE] [IF: 3.761; AIF: 1.666; IF/AIF: 2.257; Q1 (2021, InCites JCR SCIE)] [Field: N 001; N 009] [Input: 0.200]

12. **Timofejeva, Inga**; Navickas, Zenonas; Telksnys, Tadas; Marcinkevičius, Romas; Yang, Xiao-Jun; Ragulskis, Minvydas. The extension of analytic solutions to FDEs to the negative half-line // AIMS Mathematics. Springfield, MO: American institute of mathematical sciences. ISSN 2473-6988. 2021, vol. 6, iss. 4, p. 3257–3271. doi: 10.3934/math.2021195. [Science Citation Index Expanded (Web of Science); Scopus; DOAJ] [IF: 2.739; AIF: 1.666; IF/AIF: 1.644; Q1 (2021, InCites JCR SCIE)] [Field: N 001; N 009] [Input: 0.170]
13. Telksnys, T.; **Timofejeva, I.**; Navickas, Z.; Marcinkevicius, R.; Mickevicius, R.; Ragulskis, M. Solitary solutions to an androgen-deprivation prostate cancer treatment model // Mathematical methods in the applied sciences. Hoboken, NJ: John Wiley & Sons. ISSN 0170-4214. eISSN 1099-1476. 2020, vol. 43, iss. 7, p. 3995–4006. doi: 10.1002/mma.6168. [Science Citation Index Expanded (Web of Science); Scopus] [IF: 2.321; AIF: 1.944; IF/AIF: 1.193; Q1 (2020, InCites JCR SCIE)] [Field: N 001; M 001] [Input: 0.166]

ACKNOWLEDGMENTS

I would like to express my deepest gratitude to Prof. Minvydas Kazys Ragulskis, Prof. Zenonas Navickas and Prof. Romas Marcinkevičius who have provided invaluable guidance with regards to the research presented in this Thesis.

UDK 517.9+512.66.86+511.13](043.3)
SL344. 20xx-xx-xx, xx leidyb. apsk. l. Tiražas x egz. Užsakymas xxx.
Išleido Kauno technologijos universitetas, K. Donelaičio g. 73, 44249 Kaunas
Spausdino leidyklos „Technologija“ spaustuvė, Studentų g. 54, 51424 Kaunas