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FUNDAMENTALIŲJŲ MOKSLŲ FAKULTETAS

TAIKOMOSIOS MATEMATIKOS KATEDRA

Tomas Iešmantas

BAYESIAN ASSESSMENT OF RELIABILITY DYNAMICS FOR AGE-DEPENDENT SYSTEMS

Magistro darbas

Vadovas
Doc. dr. R. Alzbutas

KAUNAS, 2011

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LIST OF FIGURES

LIST OF TABLES

SUMMARY

Age-dependent highly reliable systems provide small amount of statistical information and for that reason classical frequentist methods cannot be applied due to their asymptotical assumptions. However, Bayesian methods, due to their ability to naturally couple all sources of information (including expert subjective opinions) and not rely on asymptotic assumptions, are attractive approach to solve small sample problems in age-dependent reliability modelling. In this thesis Bayesian paradigm and its applicability were presented and general methodology to analyse previously mentioned problem was obtained. Methodology successfully was applied for two real data samples: failures in European natural gas grid and electrical Instrumentation and Control components. It was concluded that presented approach is able to easily investigate small samples in nonlinear age-dependent models. Also, analysis showed that different model goodness-of-fit approaches can provide different inferences and that sometimes it can fail due to nonlinearities and heteroscedasticity present in data. For that reason Bayesian posterior model averaging procedure were applied and concluded that it gives more reliable and better calibrated results than would be in one model case. Also adaptive Metropolis superiority over classical Metropolis – Hastings algorithm for highly correlated parameters and nonlinearities in model was validated.

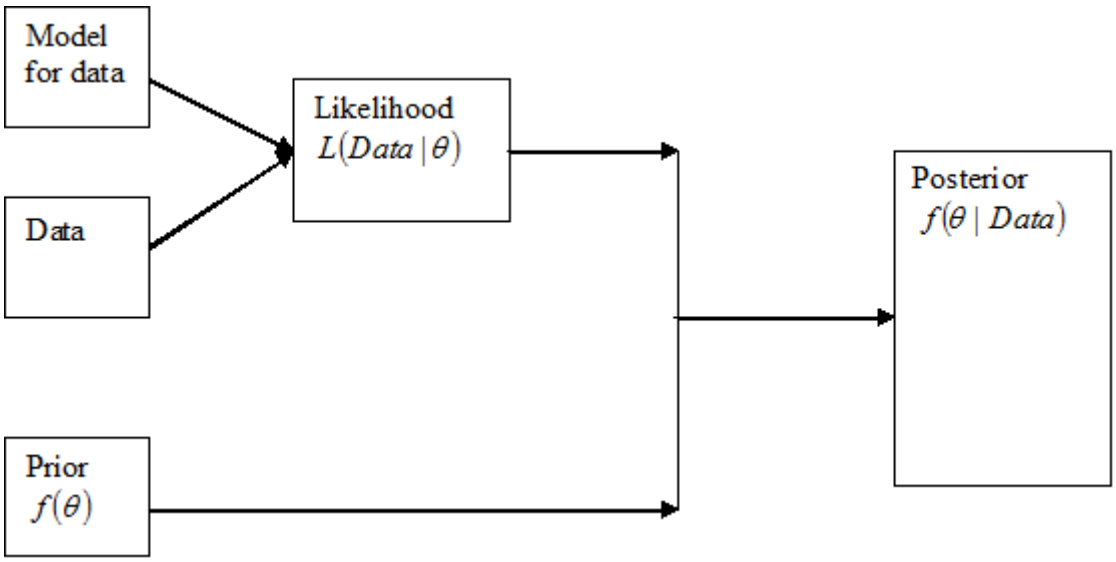
1 SANTRAUKA

Pavojingų objektų, kaip kad atominės elektrinės, cheminės gamyklos, dujų tiekimo tinklai, sauga priklauso nuo objektų amžiaus, t.y. nuo senėjimo efekto, kuris didele dalimi veikia tokių objektų sutrikimus bėgant laikui.

Senėjimo poveikis yra aktualiausias pasyvioms sistemoms ar komponentams, veikiantiems minėtuose objektuose, tačiau iki šiol dar nėra vieningos metodologijos, tinkamos nagrinėti tokioms sistemoms, kadangi gaunami statistiniai duomenys yra labai išsibarstę, reti, bei imtys mažos: klasikiniai statistiniai metodai šioms problemoms spręsti yra netinkami, dėl asimptotinių prielaidų.

Bajesinių metodų grupė suteikia galimybę mažų imčių problemą išspręsti natūraliu budų, be prielaidų, reikalaujančių daug statistinių duomenų. Šie metodai taip pat suteikia galimybę panaudoti ne tik informaciją, esančia statistiniuose duomenyse (per tikėtinumo funkciją), tačiau ir suteikia galimybę panaudoti *a priori* žinias apie nagrinėjamą sistemą apriorinių skirstinių pavidalu. Tai gali būti neinformatyvūs (psl. 18), informatyvūs (psl. 19) skirstiniai, statistiniais duomenimis grįsti skirstiniai (psl. 20), ar subjektyvi ekspertinė informacija (psl. 21).

Apriorinė informacija sujungiama su duomenimis per Bajeso formulę. Šis informacijos atnaujinimas, kurį galima suvokti kaip žinių apie sistemą būsenos atnaujinimas, gali būti pavaizduotas struktūriškai (Pav. 1.1.).



Pav. 1.1. Bajesinės procedūros schematinis vaizdas

Tokiu būdu gaunamas aposteriorinis skirstinys, nusakantis lygį žinių, apie modelio parametrus, kuomet buvo atsižvelgta į statistinius duomenis. Tuomet gautasis skirstinys yra naudojamas tiek taškiniams įverčiams gauti, tiek intervaliniams (psl. 22).

Modelio validavimui literatūroje egzistuoja labai įvairių metodų. Tai ir suderinamumo su duomenimis analizė, kuriai naudojama Bajesinė liekanų analizė, kryžminio-prognostinio patikrinimo, apriorinio-prognostinio, dalinai aposteriorinio-prognostininio patikrinimo metodikos.

Kiekvienam modeliui atskirai analizuoti taip pat taikoma taip vadinamos prognozuojančios p reikšmės (psl. 24):

$$p = P\left(D\left(y^{rep},\theta\right) > D\left(y,\theta\right) | \, y\right) = \int\int I_{\left[D\left(y^{rep},\theta\right) > D\left(y,\theta\right)\right]} f\left(y^{rep} | \, \theta\right) f\left(\theta | \, y\right) dy^{rep} d\theta ,$$

kur $D\left(y,\theta\right)$ yra skirtumų matas, apibūdinantis kiek modelio rezultatai skiriasi nuo duomenų.

Modeliams tarpusavyje palyginti yra naudojamas nuokrypio informacijos matas, apibrėžiamas taip (psl. 30):

$$DIC_i = -2\ln\left(L\left(\Theta | \, y,i\right)\right) + 2p_D .$$

Tik atskirais atvejais gaunami aposterioriniai skirstiniai yra randami analitiškai, visai kitais atvejais pasitelkiama aproksimaciniai metodai, iš kurių Markovo grandinių Monte Karlo metodų grupė yra viena labiausiai naudojamų ir išvystytų grupių Bajesinėje analizėje.

Pagrindinė idėja yra generuoti tokią Markovo grandinę (psl. 26), kurios stacionarusis skirstinys būtų ieškomas aposteriorinis skirstinys. Vienas labiausiai paplitusių metodų yra Metropolis Hastings algoritmas ir jo įvairios modifikacijos (psl. 27).

Tačiau šiame darbe yra nagrinėjama netiesiniai modeliai, kuomet parametrai yra stipriai koreliuoti ir Metropolis Hastings algoritmas praktiškai šios problemos neišsprendžia, todėl šiuo atveju svarbi metodų grupė yra adaptyvūs metodai (psl. 29). Vienas iš tokių yra adaptyvus Metropolis algoritmas, kuomet generuojama Markovo grandinė yra nehomogeninė ir nuo tam tikros iteracijos, kovariacinės matricos pavidalu, tolesni grandinės elementai generuojami atsižvelgiant į praėjusią istoriją.

Šiame darbe buvo siekiama Bajesinius metodus bei su jais susietą subjektyvios tikimybės paradigmą panaudoti sudarant metodologiją, leisiančia efektyvesnė nuo amžiaus priklausančių sistemų patikimumo analizę. Sudaryta metodologija (psl. 30) susideda iš senėjimo, kaip nuo amžiaus priklausomų žinių lygio apie sistemos parametrus apibrėžimo, tuomet pereinant prie dominančio patikimumo parametro šuolinio proceso sukonstravimo:

$$d\left(t\right) = \sum_{i=1}^{N-1} 1_{\{t_i < t < t_{i+1}\}} d\left(t_i\right) .$$

Naudojant šias konstrukcijas sudaromas duomenų modelis, kuris, kartu su statistine informacija suformuoja tikėtinumo funkciją, kuri savyje turi grynai objektyvią informaciją apie modelio parametrus, tuomet per Bajeso formule kartu su apriorine informacija gaunamas aposteriorinis skirstinys:

$$\pi\left(\Theta | \, Y,t\right) = \frac{\prod_{i=1}^{m} \pi_i\left(\theta_i\right) \prod_{j=1}^{n} f\left(y_j | \, d\left(t_j,\Theta\right)\right)}{\int_{\Omega} \prod_{i=1}^{m} \pi_i\left(\theta_i\right) \prod_{j=1}^{n} f\left(y_j | \, d\left(t_j,\Theta\right)\right) d\Theta} .$$

Turint šią struktūrą, kuri nusako žinių lygį, turima kuomet panaudojama apriorinė informacija kartu su statistine, taikomi modelio validavimo metodai, kurie jau buvo minėti aukščiau.

Taip pat, kaip universalų modeliavimo įrankį, į metodologiją įtraukiau aposteriorinio Bajesinio modelių vidurkinimo procedūrą, kuri leidžia tiek modelių validavimą, kuomet gaunamos Posteriorinės modelių tikimybės, tiek atsižvelgimą į visą informaciją, kurią savyje turi kiekvienas modelis: taip apsidraudžiama nuo potencialiai naudingo modelio atmetimo, t.y. efektyviai įvertinami neapibrėžtumai, susiję su modelio parinkimu. Jei $A(t)$ dominantis dydis, tuomet aposteriorinė suvidurkinta jo reikšmė randama taip:

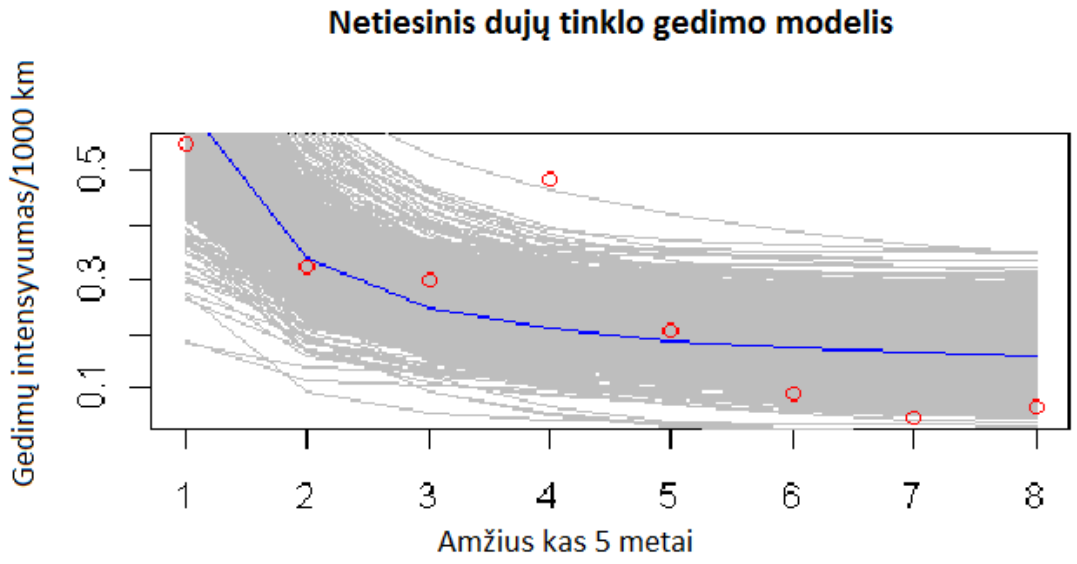
$$p\left(A(t)\,|\,Y\right)=\sum_{j=1}^{r} p\left(A(t)\,|\,Y,d_j\left(t,\Theta_j\right)\right)p\left(d_j\left(t,\Theta_j\right)\,|\,Y\right).$$

Tam, kad butu įvertintas Bajesinių metodų tinkamumas sprendžiamai problemai, buvo analizuojamos dirbtinės Puasoninės imtys su tam tikromis intensyvumų funkcijomis (psl. 33). Buvo nustatyta, kad Bajeso įvertis yra mažiau pastumtas nei kad maksimalaus tikėtinumo, kas leidžia teigti, jog Bajesiniai metodai yra tinkami mažoms imtims tirti. Be to, pastebėta, kad dažnai MTM įverčiai įgauna ekstremalias reikšmes ir jei parametrai yra apriboti intervalais, MTM įverčiai linkę koncentruotis tų intervalų kraštuose.

Metodika buvo pritaikyta dviem realioms sistemoms: elektroniniams valdymo komponentams (psl.43), kurie naudojami atominėse elektrinėse, bei gedimams Europos gamtinių dujų tinkle (psl.36).

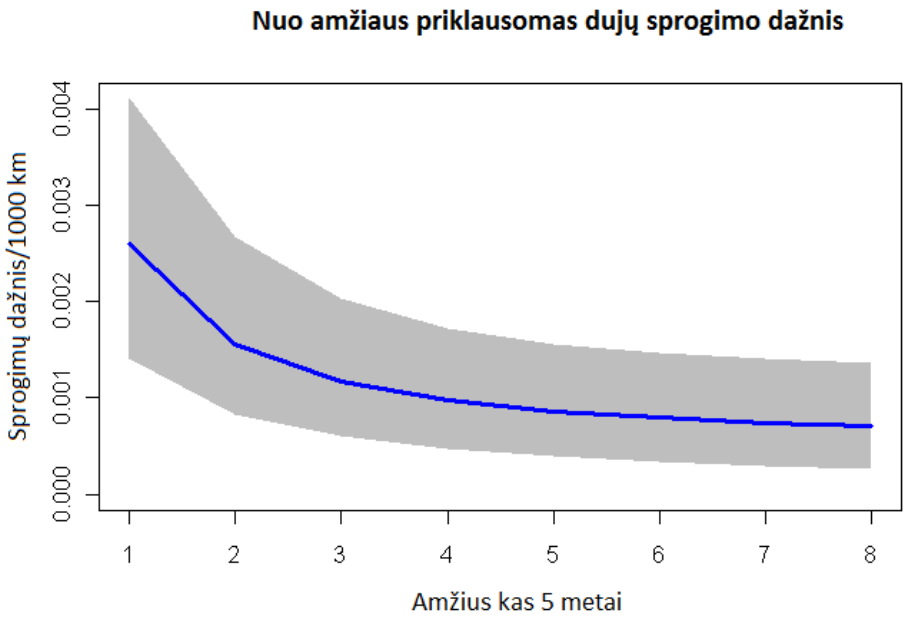
Pirmuoju atveju - Europos gamtinių dujų tinklo gedimų analizėje - gedimų intensyvumas yra mažėjantis, t.y. kuomet turimas išdeginimo periodas. Gedimų intensyvumo mažėjimas gali būti paaiškinamas tuo, kad gerėjančios technologijos, aukštesnės kokybės medžiagos, bei efektyvesnės priežiūros strategijos mažina gedimų intensyvumą.

Buvo įvertintas netiesinis intensyvumo mažėjimas bei su tuo susiję neapibrėžtumai (Pav. 1.4.).



Pav. 1.4. Europos gamtinių dujų tinklo netiesinis modelis su neapibrėžtumais

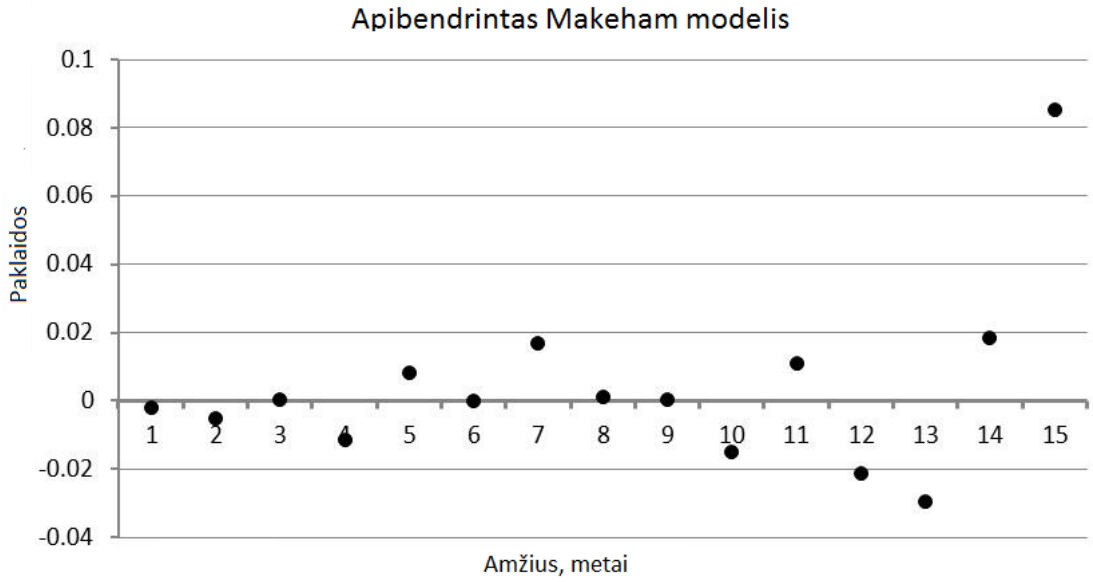
Analizuotas netiesinis modelis taip pat leido įvertinti nuo amžiaus priklausomą dujų sprogimo dažnį (Pav. 1.5.)



Pav. 1.4. Europos gamtinių dujų tinklo sprogimų dažnis

Ši analizė leidžia daug efektyviau įvertinti svarbių objektų, esančių netoli dujotiekio, sprogimų riziką bei atsižvelgiant į tai optimizuoti priežiūros darbus taip optimizuojant ekonomines sąnaudas.

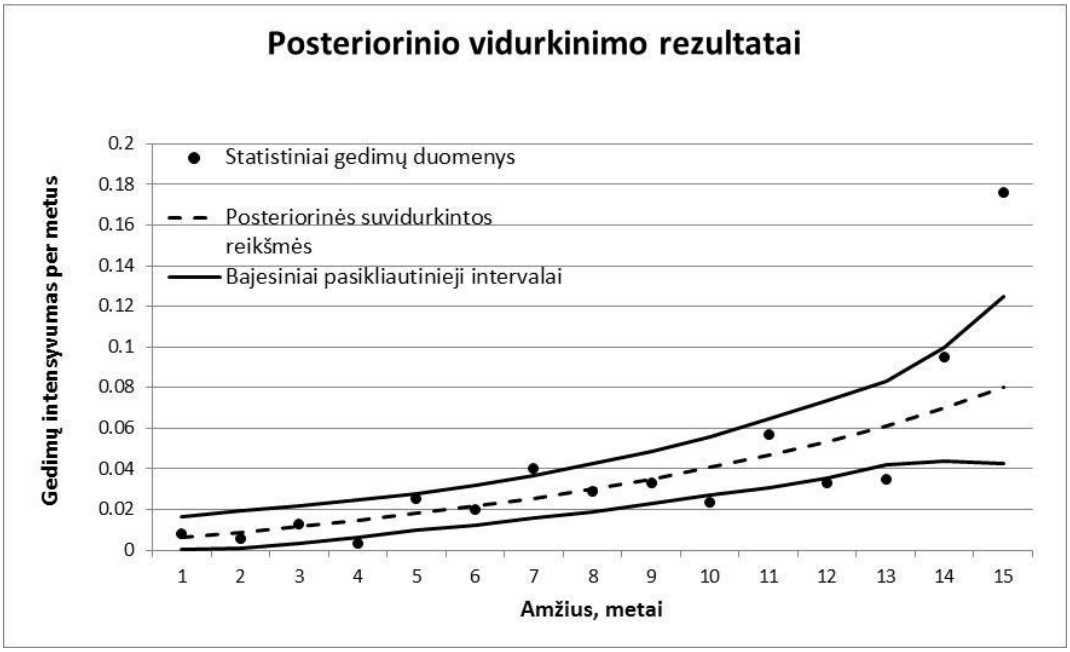
Antruoju atveju, komponentų gedimai buvo įtakojami senėjimo efekto. Atliekant duomenų analizę buvo įvertinta tokių modelio parinkimo kriterijų kaip chi kvadrato nuokrypių matas, bei deviacijos informacijos kriterijaus galimybės. Buvo nustatyta, kad chi kvadrato nuokrypių matas neveikia esant netiesiniams modeliams: gilesnės analizė parodė, kad galbūt galima priežastis yra paklaidų heteroskedastiškumas (Pav. 1.2.).



Pav. 1.2. Apibendrinto Makehamo modelio paklaidų heteroskedastiškumas

Taip pat naudotas standartinio nuokrypio matas ir deviacijos informacijos kriterijus parodė, kad skirtingi validavimo instrumentai pateikia nebūtinai sutampančias modelių aibes.

Kaip universalus sprendimas, buvo pasiūlyta naudoti aposteriorinį Bajesinį modelių vidurkinimą, kuomet neanalizuojami kiekvienas modelis atskirai, tačiau kiekvienam aibės modeliui skaičiuojamos tikimybės ir tuomet panaudojant Bajesinius modelius gaunamos posteriorinės suvidurkintos reikšmės (pav. 1.3.).



Pav. 1.3. Posteriorinės suvidurkintos reikšmės bei 0.95 tikimybės intervalai

Taip pat šios analizės metu buvo analizuotas adaptyvus Metropolis algoritmas ir palygintas su Metropolis-Hastings algoritmu. Nustatyta, kad adaptyvus Metropolis algoritmas yra tinkamas netiesiniams modeliams bei itin koreliuotiems parametrams analizuoti, kadangi konvergavimas į stacionarų skirtinį pasiekiamas daug efektyviau bei greičiau.

Apibendrinant galima pasakyti, kad šiame darbe aš pristačiau bendra metodologiją, panaudojant Bajesinius metodus, nuo amžiaus priklausomo patikimumo analizei. Buvo parodyta, kad metodologija gali būti naudojama mažoms imtims, išsibarsčiusiems duomenims bei netiesiniams modeliams su daug parametrų (Makeham, Xie ir Lai dėsniai) nagrinėti.

Pasiūlyta metodologija buvo pritaikyta elektrinių kontrolės komponentų senėjimo analizėje. Ši analizė buvo atlikta naudojant dalimis homogeninį Puasono modelį su keletu gedimų intensyvumo funkcijų.

Analizuojant bei validuojant modelius, buvo pastebėta, kad nei viena iš modelio parinkimo technikų negali duoti vienareikšmiško atsakymo. Prognozuojančios p reikšmės gali būti klaidinančios ir gali arba visai neturėti praktinės reikšmės (chi kvadrato mato atvejis), arba gali pasiūlyti daugiau nei vieną modelį (standartinio nuokrypio mato atvejis). Kaip vieną iš galimų paaiškinimų, kodėl chi

kvadrato matu paremtos prognozuojančios p reikšmės netinka netiesiniu atveju, pasiūliau paklaidų heteroskedastiškumo aiškinimą.

Deviacijos informacijos kriterijus taip pat gali pateikti daugiau nei vieną modeli (ir nebūtinai modelių aibė sutaps su p reikšmių pasiūlyta aibe). Dėl šių priežasčių, egzistuoja didelė tikimybė atmesti modelį, kuris taip pat gali pateikti gerus rezultatus. Taigi, modelio parinkimas turi būti atliktas labai atsargiai. Verta paminėti, kad ir kiti modelio parinkimo kriterijai, kaip kad Bajesinis informacijos kriterijus, Bajesiniai faktoriai ir kt. turi aukščiau įvardintus trūkumus.

Tam, kad išvengti modelio parinkimo procedūrų trūkumų, Bajesinė aposteriorinė modelių vidurkinimo procedūra buvo pritaikyta visai analizuojamai modelių aibei. Toks suvidurkinimas duoda geresnius rezultatus, geriau kalibruotas prognozes, kadangi atsižvelgiama ne tik į neapibrėžtus įnešamus dėl duomenų išsibarstymo, bet ir į neapibrėžtumus iškylančius dėl modelio parinkimo.

Nepaisant visų Bajesinio vidurkinimo privalumų, ši metodika taip pat turi tam tikrų trūkumų: procedūra negali analizuoti begalinės aibės modelių ir kuomet pasirenkama galimų modelių aibė, tinkamas modelis gali būti taip ir neįtrauktas į tą aibę ir Bajesinio vidurkinimo procedūra neturi jokių saugiklių leidžiančių tą pastebėti.

Ši analizė gali būti naudojama kaip pagrindas tolesnei senėjančių sistemų, struktūrų ir komponentų analizei. Jos bendrumas ir idėja, kad senėjimas ar degradacija gali būti laikoma kaip nuo amžiaus priklausantys įsitikinimai apie sistemos patikimumo parametrus, leidžia plataus spektro problemų analizę: tai gali būti stochastinė trūkio augimo elgsena, tai gali būti ir degradacijos modeliavimas panaudojant Markovo būsenas ir t.t.

Analizuojant Europos gamtinių dujų tinklo patikimumą, buvo pateikta naujausia statistinė informacija apie dujotiekio tinklo gedimus ir panaudota nuo amžiaus priklausančių gedimo intensyvumo bei sprogimų dažnio analizėje.

Bajesiniai metodai leido robastišką gedimų intensyvumo funkcijos parametrų įvertinimą; dar daugiau, parametrų neapibrėžtumai buvo įvertinti ir panaudoti skaičiuojant pasikliautinuosius intervalus, kurie yra daug lengviau interpretuojami nei kad klasikinėje statistikoje, kas leidžia inžinieriams efektyviau panaudoti statistinius metodus.

Įvertintas amžiaus momentas, kuomet gedimų intensyvumas pasieks pastovią reikšmę (tam tikros paklaidos ε ribose) parodė, kad nėra būtinybės atsisakyti intensyvumo funkcijos ir pereiti prie pastovios reikšmės.

Nuo laiko priklausomas gedimų intensyvumas yra daug priimtinesnis dujotiekio tinklo priežiūros strategijoms vystyti, taip pat įvertinant riziką skirtinguose tinklo taškuose: tai gali būti atlikta pasinaudojant pateikta nuo amžiaus priklausomu dujų sprogimo dažniu.

2 INTRODUCTION

Safety of energy facilities, chemical factories, oil companies, etc. in many cases depends on their components reliability, which is mainly age-dependent. Unnoticed on time, ageing effect can cause failures or multiple damages at given non-standard operating conditions or breakdown situations. Equipment ageing is caused by two important impacts: by operating conditions and technical inspection actions.

Systems can be distinguished into two main categories: active and passive. The term "passive system" identifies a system, which is composed entirely of passive components and structures, or a system, which uses active components in a very limited way to initiate subsequent passive operation [33].

The passive components (e.g. heat exchangers, pipes, vessels, electrical cables, structures, etc.) are usually neglected or not modelled implicitly in the risk assessment models as having very low failure probability, but they could have an increasing contribution due to ageing effects.

Age-dependent reliability study requires more data and more extended models than a usual reliability analysis. With regard to data, one basic issue is scattering of failure histories for passive components and systems. Because of this scattering of failure data, reliability and risk model parameters, which are estimated from the raw data, have large associated uncertainties.

Usually, passive systems, components do not provide large samples of failure data and because of lack of data it is extremely difficult to deal with degradation of passive systems/components by using classical statistical methods which requires considerable amount of data. However, uncertainties, related to data scattering could be reduced and assessment of age-dependent reliability of passive systems could be improved by considering, so called, prior information – experience of other similar facilities, subjective expert insights. Then, by the use of available statistical data, prior knowledge can be revised by Bayes formula [7]. When data arrives, Bayes theorem tells how to move from prior beliefs to new conditional probabilities for the quantities of interest.

The main advantage of the Bayesian approach is that it can rely on multiple sources of evidence including: warranty data, customer research surveys, proving ground test data, etc. It also has the potential to systematically quantify and process "soft" evidence such as expert knowledge [27].

In addition to their ability to deal with sparse data, Bayesian techniques are appropriate for use in PRA (Probabilistic Risk Assessment) because they are derived from the framework of subjective probability [30]. Further, as Siu and Kelly noticed in their paper [30], practical advantage of the subjective probability framework in PRA applications is that propagation of uncertainties through complex models is relatively simple. On the other hand, it is very difficult, and intractable in "real" problems, to propagate classical statistical confidence intervals through PRA models to estimate a confidence interval for a composite result of interest.

Despite of the advantages offered by Bayesian methods, applicability of it was very limited and generally confined by the use of so called conjugate prior distributions, which provides analytically tractable problem solutions just for quite unsophisticated applications.

The advent of Markov Chain Monte Carlo (MCMC) sampling has proliferated Bayesian inference throughout the world, across a wide array of disciplines [10]. MCMC methods are a class of algorithms for sampling from probability distributions based on constructing a Markov chain that has the desired distribution as its equilibrium distribution [40, 41]. The state of the chain after a large number of steps is then used as a sample from the desired distribution. MCMC algorithms enabled analysis of highly complex Bayesian models. The freely available software package known as Bayesian inference Using Gibbs Sampling (BUGS) has been in the vanguard of this proliferation since the mid-1990s [28].

Bayesian methods allow data to be combined with ‚prior' nformation to produce a posterior distribution for parameters. This posterior is used to quantify uncertainty about the parameters and functions of parameters.

Combination of extensive past experience and physical/chemical theory can provide prior information to form a framework for inference and decision making. In many applications it is necessary to combine prior information with limited additional observational or experimental data. [43]

The use of bayesian methods as both an information combination scheme and an updating tool has become widespread, combining or updating prior information with existing information about events. Bayesian methods stem from the application of Bayes' theorem in probability. Bayesian methods provide ways of handling various kinds of uncertainties.

In Bayesian paradigm, uncertainty is quantified in terms of a personal or subjective probability following the axioms of probability theory. There are many uses for and interpretations of Bayes' theorem extolling its virtues. A few are summarized below [38]:

1. Bayes' theorem indicates how point estimates (and their associated uncertainties) are updated (combined) in light of additional pertinent information or data (such as relevant information from computer models);
2. Bayes' theorem is a statistical method for combining different kinds of data and/or information about some quantity of interest (such as reliability of a system);
3. Bayes' theorem describes how uncertainties in data regarding a quantity of interest (such as a performance measure of a system) are modified in light of other available information about the quantity of interest;
4. Bayes' theorem provides a mechanism for inverting conditional probability distributions of data in light of additional prior information and data.

The difficulty in using Bayes‘ theorem is the determination of what information should be labelled as prior and what should be labelled within the likelihood function. Because the likelihood does not need to satisfy the axioms of probability, it is often misunderstood or misspecified. According to the theory, the likelihood should be the data or information collected from an experiment, or observation, or new information gained since collection of the old (prior) information. The most prominent argument brought against Bayesian-based methods by those with the more frequentist view is for the use of the subjectivist or degree of belief probability theory required to interpret the meaning of the prior and posterior probabilities.

On the other hand, the frequentist interpretation of probability violates some basic principles of that theory. Frequentists cannot accommodate any existing historical information, ant they rely solely on experimental or observational data that may be too sparse (as in the case of passive system failure data) for formulating conclusions.

3 MATHEMATICAL BACKGROUND FOR RELIABILITY ASSESSMENT

The reliability characteristics of an item can be quantified and measured in variety ways. The reliability of a product can be calculated during its design and development using probabilistic techniques.

There are three fundamental reliability figures of merit, namely the probability or likelihood of successful performance over time, the expected or mean time of successful performance, and the failure rate.

The reliability of an item is the probability that it will, in the future, perform its intended functions under specified conditions for a specified period of time. If the random variable T is defined as the time to failure of an item, then its reliability is expressed in:

$$R(t) = \int_{t}^{\infty} f(y)dy \qquad \textbf{(3.1)}$$

where f(y) is the probability density function of T.

The properties of the reliability function can be summarized as follows:

- R is a non – increasing function of time
- R is bounded: $0 \le R(t) \le 1$;
- $R(0) = 1$;
- $\lim_{t\to\infty} R(t) = 0$.

The failure rate of a product is the instantaneous rate of failure as a function of time, usually defined as the age of each item starting at zero time. The instantaneous failure rate of a product is also known as hazard rate. The failure rate is a measure of susceptibility to failure as a function of time.

The failure rate at time t, denoted by $\lambda(t)$, is defined to be the number of failures occurring in the time interval Δt, denoted by $n_f(t+\Delta t)$, divided by the number of survivors at time t, denoted by $n_s(t)$, as Δt approaches zero:

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{n_f(t+\Delta t)}{n_s(t)\Delta t} \tag{3.2}$$

Or in terms of reliability function:

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{R(t) - R(t+\Delta t)}{R(t)\Delta t} = -\frac{1}{R(t)} \frac{d}{dt} R(t) = \frac{f(t)}{R(t)} \tag{3.3}$$

The properties of failure rate:

- $\lambda(t) \geq 0, \forall t \in [0; +\infty)$
- $\lim_{t \to \infty} \int_0^t \lambda(y)\, dy = \infty$

Most of system reliability quantities are expressed in terms of failure rate $\lambda(t)$, so this parameter is one of the most important parameter in reliability theory.

4 BAYESIAN MODELLING AND INFERENCE

4.1 BAYES' RULE AND PRIOR INFORMATION UPDATING

In order to make probability statements about θ given y, we must begin with a model providing a joint probability distribution for θ and y. The joint probability mass or density function can be written as a product of two densities that are often referred to as the prior distribution $f(\theta)$ and the sampling distribution $f(y \mid \theta)$ respectively [5]:

$$f(\theta, y) = f(\theta) f(y \mid \theta) \tag{4.1}$$

Bayes's rule provides a mechanism for combining prior information with sample data to make inferences on model parameters. This is illustrated in Figure 4.1.1

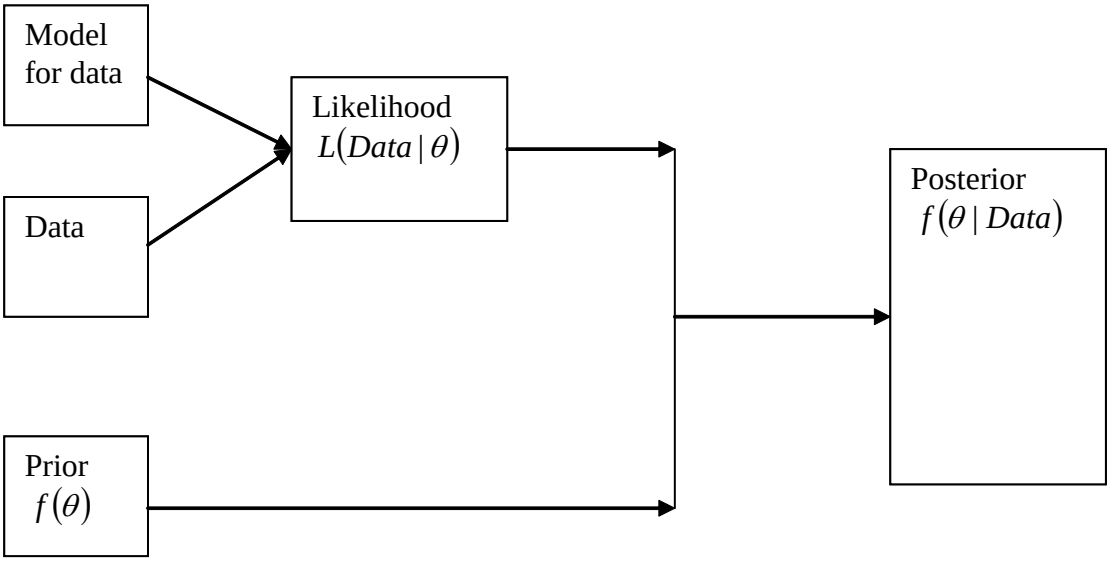


Figure 4.1.1 Bayesian methods for making inferences or predictions

Analytically, for a vector of parameters θ the procedure is as follows. Prior knowledge about θ is expressed in terms of a pdf denoted by $f(\theta)$. The likelihood for the available data and specified modelį is given by $L(Data \mid \theta) = L(\theta; Data)$. Then, using Bayes‘s rule, the conditional distribution of θ given the data provides the posterior pdf of θ, representing the updated state of knowledge about θ. This posterior distribution can be expressed as

$$f(\theta \mid Data) = \frac{L(Data \mid \theta) f(\theta)}{\int L(Data \mid \theta) f(\theta) d\theta} \quad \textbf{(4.2)}$$

In general, it is impossible to compute the integral in denominator in closed form. Numerical methods are needed and these methods can be computationally intensive when the length of θ is more than two or three. In the past this has been an impediment to the use of Bayesian methods. Today, however, new statistical and numerical methods that take advantage of modern computing power are making it feasible to apply Bayesian methods to a much wider range of real problems [43].

4.2 PRIOR INFORMATION

It is convenient to divide available prior information about a parameter into three different categories [43]:

1. Parameters that are given as known, leading to a degenerate prior distribution;
2. Parameters with a diffuse or approximately non-informative prior distribution;
3. An informative, non-degenerate prior distribution.

In general, there are two possible sources or prior information: (1) expert or other subjective opinion or (2) past data. The prior pdf $f(\theta)$ may be either informative or not. Loosely speaking, a non-informative prior distribution is one that provides little or no information about any of the parameters in θ. Such a prior distribution is useful when it is desired to let the data speak for themselves without being influenced by previous data, expert opinion, or other available prior information.

4.2.1 NON INFORMATIVE (DIFFUSE) PRIOR DISTRIBUTIONS

Non informative (or approximately non informative) prior pdfs are constant over the range of the model parameters. Some non-informative priors are “improper” because they do not integrate to a finite quantity. Improper pdfs cause no difficulties as long as the resulting posterior pdf is “proper”.

Whether this is so or not depends on the form of the model and the available data. „Proper“ distributions are called *diffuse distributions*.

While there are a number of formalism for developing non informative prior distributions, one of the most common uses *Jeffrey's rule*, which results in a distribution often called a *Jeffreys' prior*. Suppose that we have a one-to-one transformation of our parameter $\phi = h(\theta)$. There are two ways we can think about determining a prior distribution for ϕ. One is to use a rule to determine a prior distribution $f(\theta)$ for θ and to use the change of variables technique to determine the implied prior distribution for ϕ; the second is to use the same rule to directly determine a prior distribution for ϕ. Jeffreys' rule states that any rule for determining a prior distribution should yield the same prior distribution for ϕ whether we transform from a prior on θ or determine a prior directly for ϕ [29].

Define the *expected Fisher information* as

$$I(\theta) = -E_{\theta}\left(\frac{d^2 \ln(f(y \mid \theta))}{d\theta^2}\right) \qquad \textbf{(4.3)}$$

Jeffreys' rule defines a non-informative prior as $f(\theta) \propto (I(\theta))^{\frac{1}{2}}$.

Table 4.2.1 summarizes common choices for non-informative prior distributions [29].

Table 4.2.1 Common choice for non-informative prior distributions

Parameters	Non informative prior
$Binomial(\pi)$	$Beta(0.5,0.5)$
$multinomial(\pi)$	$Dirichlet(0.5,0.5,...,0.5)$
$Poisson(\lambda)$	$\lambda^{-\frac{1}{2}}$
$Normal(\mu, \sigma \text{ known})$	constant k
$Normal(\sigma, \mu \text{ known})$	σ^{-1}

Prior distributions can also be based on available data. Combining past data with a non-informative prior distribution gives a posterior pdf that is proportional to the likelihood. This posterior pdf can then serve as a prior pdf for further updating with new data.

4.2.2 INFORMATIVE PRIOR DISTRIBUTIONS

We use informative prior distributions when we have information about the parameters of our model before we collect data. In reliability problems, there are six broad sources of information for constructing informative prior distributions [29]:

1. physical/chemical theory;

2. computational analysis;
3. previous engineering and qualification test results from a process development program;
4. industry wide generic reliability data;
5. past experience with similar devices;
6. expert opinion.

There are numerous industry wide generic sources of reliability data reported in a variety of media, such as reliability databases or reliability data handbooks. These sources report the results of analyses performed on actual failure or maintenance event data or, in some cases, are based on expert opinion. They usually contain component failure probabilities, failure rates, and, in some cases, initiating event frequencies.

Expert judgment is often used in assessing a prior distribution. In assessing probability distribution based on expert opinion there are many potential biases that have been identified that either must be minimized or, at the very least, accounted for when assessing prior probability distributions.

Several heuristics in connection with developing an informative prior distribution [29]:

1. Beware of zero values. If the prior distribution says that a value of the parameter is impossible, than no quantity of data can overcome this;
2. when using expert opinion, beware of cognitive biases caused by the way people think;
3. beware of generating overly narrow prior distributions.
4. ensure that the information used to generate the prior distribution is relevant to the problem at hand;
5. be careful when assessing prior distributions on parameters that are not directly observable;
6. beware of conservatism. Realism is the desired ideal, not conservatism.

4.2.3 DATA-BASED INFORMATIVE PRIORS

Defining the prior and likelihood functions correctly is critical in applying Bayesian methods in risk and reliability analysis. In particular, the more precise and accurate the prior can be made based on past data, the more precise the result of the analysis will be, especially for situations with little data.

A number of different methods have been proposed in the literature for formulating an informative prior based on past data, and each of these methods has different theoretical underpinnings. Methods, proposed in [35] is (1) the method of moments, (2) maximum likelihood, (3) maximum entropy estimation, (4) two-stage updating of a non-informative “pre-prior”, (5) a credible interval-based method.

Large data samples usually are not the case in failure analysis. However, bootstrap estimation cal help to overcome this difficulty. As discussed in [37], main idea of bootstrap method is resampling from already observed sample (original sample). Resample's from original sample represents what we would get if we took many samples from the population.

Overview of data-based informative prior construction methods can be found in [35]. Here two-stage updating of a non-informative "pre-prior" will be considered, because in ageing analysis, bootstrap is not appropriate method and resampling is not possible – information about the age when failure occurred can be lost.

A simple way to formulate an informative prior based on past data is to treat the past data as a separate data sample to be used in updating an appropriate prior that incorporates neither the past data nor the new data. This would then form a posterior which would become the prior for the analysis using the new data. Initial prior is here termed as "pre-prior". With this approach, the new data is effectively treated as an extension of the past data set, and the past data is assumed to dominate the pre-prior so that the pre-prior has little effect on the results.

4.2.4 EXPERT OPINION

The elicitation of a prior distribution for a single parameter may be straightforward if there has been considerable experience in estimating or observing estimates of that parameter in similar situations. For a vector of parameters, however, the elicitation and specification of meaningful joint prior distribution is more difficult. In general, marginal distributions for individual parameters do not completely determine the joint distribution. Also, it is difficult to elicit opinion on dependences among parameters and then express these as a joint distribution. Also, it may not be reasonable to elicit opinion about parameters from a standard parameterization when those parameters have no physical or practical meaning [43].

A general approach is to elicit information about particular quantities that, from past experience, can be specified approximately independently. Then the corresponding prior distribution for these quantities can be described as being approximately independent.

When there is useful informative prior information for a parameter, one elicits a general shape or form of the distribution and the range of it.

4.3 THE USE OF POSTERIOR DISTRIBUTION FOR ESTIMATION

4.3.1 BAYESIAN POINT ESTIMATION

Bayesian inference for θ and functions of the parameters $g(\theta)$ are entirely based on the posterior pdf's $f(\theta \mid y)$ and $f(g(\theta) \mid y)$. If $g(\theta)$ is a scalar, a common Bayesian estimate of $g(\theta)$ is the mean of the posteriors distribution which is given by

$$\hat{g}(\theta) = E[g(\theta) \mid y] = \int g(\theta) f(\theta \mid y) d\theta \qquad \textbf{(4.4)}$$

This estimate of $g(\theta)$ is the Bayesian estimate that minimizes the square error loss. Other possible choices to estimate $g(\theta)$ include the mode of posterior pdf and the median. Such estimates are easy to compute from a simulated sample from a posterior. In particular [43],

$$\hat{g}(\theta) \approx \frac{1}{M} \sum_{i=1}^{M} g(\theta_i) \qquad \textbf{(4.5)}$$

is the sample mean.

4.3.2 BAYESIAN INTERVAL ESTIMATION

A $100(1-\alpha)\%$ Bayesian lower confidence bound (credible bound) for a scalar function $g(\theta)$ is value $\underset{\sim}{g}$ satisfying $\int_{\underset{\sim}{g}}^{\infty} f[g(\theta) \mid y] dg(\theta) = 1-\alpha$. Upper bound is value $\tilde{g}$ satisfying

$$\int_{-\infty}^{\tilde{g}} f[g(\theta) \mid y] dg(\theta) = 1-\alpha .$$

4.4 THE PREDICTIVE DISTRIBUTION AND MODEL CHECKING

4.4.1 PREDICTION WITHIN BAYESIAN FRAMEWORK

In Bayesian theory, predictions of future observables are based on ***predictive distributions***, that is, the distribution of the data averaged over all possible parameter values. For this reason, when data ***y*** have not been observed yet, predictions are based on the marginal likelihood

$$f(y) = \int f(y \mid \theta) f(\theta) d\theta \qquad \textbf{(4.6)}$$

which is the likelihood averaged over all parameter values supported by our prior beliefs. Hence, $f(y)$ is also called ***prior predictive distribution***.

Usually, after having observed data y, one finds the prediction of future data y' more interesting. Following this logic, we calculate the ***posterior predictive distribution***

$$f(y' \mid y) = \int f(y' \mid \theta) f(\theta \mid y) d\theta \quad \textbf{(4.7)}$$

which is the likelihood of the future data averaged over the posterior $f(\theta \mid y)$.

By using the predictive distribution, we can quantify our knowledge about future as well as measure the probability of again observing in the future each y_i assuming that the adopted model is true. For this reason, we may use the predictive distribution not only to predict future observations but also to construct goodness-of-fit diagnostics and perform model checks fore ach model‘s structural assumptions [24].

If the model fits, then replicated data generated under the model should look similar to observed data. To put it another way, the observed data should look plausible under the posterior predictive distribution. This is really a self-consistency check: an observed discrepancy can be due to model misfit.

Basic techniques for checking the fit of a model to data, is to draw simulated values from the posterior predictive distribution of replicated data and compare these samples to the observed data. Any systematic differences between the simulations and the data indicate potential failings of the model [5].

4.4.2 MODEL EVALUATION AND CHECKING

Model checking, or assessing the fit of a model, is a crucial of any statistical analysis. Before drawing any firm conclusions from the application of a statistical model to a data set, an investigator should assess the model’s fit to make sure that the important features of the data set are adequately captured. Serious misfit (failure of the model to explain a number of aspects of the data that are of practical interest) should result in the replacement or extension of the model, if possible.

In Bayesian statistics, a researcher can check the fit of the model using a variety of strategies: (1) checking that the posterior inferences are reasonable, given the substantive context of the model; (2) examining the sensitivity of inferences to reasonable changes in the prior distribution and the likelihood; (3) checking that the model can explain the data, or in other words, that the model is capable of generating data like the observed data [11].

4.4.3 MODEL CHECKING OVERVIEW

The first and most natural form of model checking is to check that the posterior inferences are consistent with any information that was not used in the analysis. This could be data from another data source that is not being considered in the present analysis or prior information that was not incorporates in the model. Nonsensical or paradoxical parameter values in the posterior distribution may indicate a problem in the computer program used to carry out the analysis. Implausible results

may also indicate that a prior distribution chosen for mathematical or computational convenience is inappropriate.

Sensitivity analysis is important because any single model will tend to underestimate uncertainty in the inferences drawn. Other reasonable models could have fit the data equally well yet yielded different inferences. Model averaging is one approach to taking account of this information. From the model checking perspective the existence of other reasonable models points to the need for sensitivity analysis. The basic technique of sensitivity analysis is to fit several probability models to the same data set, altering either prior distribution, the likelihood, or both, and studying how the primary inferences for the problem at hand change.

4.4.4 MODEL CONSISTENCY WITH THE DATA

It seems that a minimum requirement for a good probability model would be that it is able to explain the key features in the data set adequately, or put differently, that data generated by the model should look like the observed data. This is a self-consistency check.

There are a number of approaches for carrying out a self-consistency check of the model: (1) Bayesian residual analysis; (2) cross-validatory predictive checks; (3) prior predictive checks; (4) posterior predictive checks; (5) partial posterior predictive checks; (6) repeated data generation and analysis. These approaches can be found here [25, 12,12,34] in more detailed form.

However, in next chapter posterior predictive check approach is presented in more comprehensive way. Posterior predictive model checks are straightforward to carry out once the difficult task of generating simulations from the posterior distribution of the model parameters is done. One merely has to take the simulated parameter values and then simulate data according to the model's sampling distribution to obtain replicated data sets.

4.4.5 POSTERIOR PREDICTIVE MODEL CHECKING TECHNIQUES

Let $f(y|\theta)$ denote the sampling or data distribution for a statistical model, where θ denotes the parameters in the model. Let $f(\theta)$ be the prior distribution on the parameters. Then the posterior distribution of θ is $f(\theta|y) = \dfrac{f(y|\theta)\,f(\theta)}{\int_{\Omega} f(y|\theta)\,f(\theta)\,d\theta}$. Let y^{rep} denote replicate data the one might observe if the process that generated the data y is replicated with the same value of θ that generated the observed data. Then y^{rep} is governed by the *posterior predictive distribution* (or the predictive distribution of replicated data conditional on the observed data),

$$f(y^{rep}|y) = \int f(y^{rep}|\theta)\,f(\theta|y)\,d\theta \qquad \textbf{(4.8)}$$

To carry out model checks test *quantities* or *discrepancy measures* $D(y,\theta)$ are defined [1], and the posterior distribution of $D(y,\theta)$ compared to the posterior predictive distribution of $D(y^{rep},\theta)$, with any significant difference between the indicating a model failure. If $D(y,\theta)=D(y)$, then the discrepancy measure is a test statistic in the usual sense.

Model checking can be carried out by graphically examining the replicate data and the observed data, by graphically examining the joint distribution of $D(y,\theta)$ and $D(y^{rep},\theta)$, or by calculating a numerical summary of such distribution. One numerical summary of the model diagnostic‘s posterior distribution is the tail-area probability or as it is sometimes known, the posterior predictive p-value:

$$p = P\left(D\left(y^{rep},\theta\right) > P\left(y,\theta\right) | \, y\right) = \int\int I_{\left[D\left(y^{rep},\theta\right) > D\left(y,\theta\right)\right]} f\left(y^{rep} | \, \theta\right) f\left(\theta | \, y\right) dy^{rep} d\theta \qquad \textbf{(4.9)}$$

Though the tail area probability is only one possible summary of the model check it has received a great deal of attention. The inner integral in ((4.9) can be interpreted as a traditional p-value for assessing a hypothesis about a fixed value of θ given the test measure D. If viewed in this way, the various model checking approaches represent different ways of handling parameter θ. The posterior predictive p-value is an average of the classical p-value over the posterior uncertainty about the true θ.

Meng [44] provides a theoretical comparison of classical and Bayesian p-values. One unfortunate result is that posterior predictive p-values do not share some of the features of the classical p-values that dominate traditional significance testing. In particular, they do not have a uniform distribution when the assumed model is true, instead they are more concentrated around 0.5 than a uniform distribution.

Because posterior predictive checks are Bayesian by nature, a question arises about the sensitivity of the results obtained to the prior distribution on the model parameters. Because posterior predictive checks are based on the posterior distribution they are generally less sensitive to the choice of prior distribution than are prior predictive checks. More failures are detected only if the posterior inferences under the model seem flawed. Unsuitable prior distributions may still be judged acceptable if the posterior inferences are reasonable.

Strongly informative prior distributions may of course have a large impact on the results of posterior predictive model checks. The replicated data sets obtained under strong incorrect prior specifications may be quite far from the observed data. In this way posterior predictive checks maintain the capability of rejecting probability model if the prior distribution is sufficiently poorly chosen to negatively impact of the model to the data [11].

Technically, any function of the data and the parameters can play the role of a discrepancy measure in posterior predictive checks. The choice of discrepancy measures is very important. Virtually all models are wrong, and a statistical model applied to a data set usually explains certain aspects of the data adequately and some others inadequately. The challenge to the researcher in model

checking is to develop discrepancy measures that have the power to detect the aspects of the data that the model cannot explain satisfactorily. A key point is that discrepancy measures corresponding to features of the data that are directly addressed by model parameter will never detect a lack of fit. Discrepancy measures that relate to features of the data not directly addressed by the probability model are better able to detect model failures.

As noted in [24], posterior p-values can be used for checking the structural assumptions of the fitted model. For example, we can check whether the skewness and the kurtosis of the predictive and actual data are in agreement (which is particularly useful for normal models), or whether the assumption of equal mean and variance in Poisson models is valid.

For Poisson models we can calculate the sample dispersion index $DI(y) = \frac{\bar{y}}{SD^2(y)}$ and its corresponding p-value

$$p_{DI} = P\left(DI\left(y^{rep}\right) > DI(y)\right) \qquad \textbf{(4.10)}$$

A critical shortcoming of posterior-predictive p values is that they are not (even asymptotically) uniformly distributed. That is, the presumed sampling distributions of discrepancy measure are not actually achieved in posterior-predictive simulations. Although this fact does not preclude the use of this methodology for performing case diagnostics, it severely limits its application for formal model assessment [39].

5 NUMERICAL METHODS FOR BAYESIAN INFERENCE

5.1 MONTE CARLO METHODS

Monte Carlo methods are a class of computational algorithms that rely on repeated random sampling to compute their results. Monte Carlo methods are often used in simulating physical and mathematical systems. Because of their reliance on repeated computation of random or pseudo-random numbers, these methods are most suited to calculations by a computer and tend to be used when it is infeasible or impossible to compute an exact result with a deterministic algorithm.

5.2 MARKOV CHAINS

Suppose we generate a sequence of random variables, $\{X_0, X_1, X_2, ...\}$, such that at each time $t \geq 0$, the next state X_{t+1} is sampled from a distribution $P(X_{t+1} \mid X_t)$ which depends only on the current state of the chain, X_t. That is, given X_t, the next state X_{t+1} does not depend further on the history of the chain $\{X_0, X_1, ..., X_{t-1}\}$. This sequence is called a *Markov chain*, and $P(.|.)$ is called the *transition kernel* of the chain. We will assume the chain is time-homogenous: that is, $P(.|.)$ does not depend on t.

Subject to regularity conditions, the chain will gradually "forget" its initial state and $P^{(t)}(.\mid X_0)$ will eventually converge to a unique *stationary* distribution, which does not depend on t or X_0. Denote the stationary distribution by $\phi(.)$. Thus as t increases, the sampled points $\{X_t\}$ will look increasingly like dependent samples from $\phi(.)$.

Thus, after a sufficiently long *burn-in* of say m iterations, points $\{X_t; t = m+1,...,n\}$ will be dependent samples approximately from $\phi(.)$. Now the output from Markov chain can be used to estimate the expectation $E[f(X)]$, where X has distribution $\phi(.)$. Burn in samples are usually discarded for this calculation, giving an estimator

$$\overline{f} = \frac{1}{n-m}\sum_{t=m+1}^{n} f(X_t) \qquad \textbf{(5.1)}$$

This is called an *ergodic average*. Convergence to the required expectation is ensured by the ergodic theorem [26]. More profound and comprehensive analysis of Markov chains can be found in [26,41]

5.3 METROPOLIS-HASTINGS ALGORITHM

One problem with applying Monte Carlo (MC) integration is in obtaining samples from some complex probability distribution $p(x)$.

Suppose our goal is to draw samples from some distribution $p(x)$. The Metropolis algorithm than generates a sequence of draws from distribution is as follows:

1. Start with any initial value $X_0 : f(X_0) > 0$
2. Using X_0 value, sample a candidate point X^* from some jumping distribution $q(X_1 \mid X_2)$, which is the probability of returning a value of X_2 given previous value of X_1. This distribution is also reffered to as the proposal or candidate-generating distribution. The only restriction is that it is symmetric $q(X_1 \mid X_2) = q(X_2 \mid X_1)$.
3. Given the candidate point X^*, calculate the ratio of the density at the X^* and current X_{t-1} points:

$$\alpha = \frac{p(X^*)}{p(X_{t-1})}$$

4. If the jump increases the density $(\alpha > 1)$, accept the candidate (set $X_t = X^*$) and return to step 2. If the jump decreases the density, then with probability α accept the candidate points, else reject it and go to step 2.

This generates Markov chain $(X_0, X_1, ..., X_k, ...)$, as the transient probabilities from X_t to X_{t+1} depends only on X_t

Following a sufficient burn-in period (say ***k*** steps), the chain approaches its stationary distribution and samples from vector $(X_{k+1}, X_{k+2}, ..., X_{k+n})$ are samples from $p(x)$.

Hastings generalized the Metropolis algorithm by using an arbitrary transition probability function $q(X_{t+1} \mid X_t) = P(X_t \to X_{t+1})$ and setting the acceptance probability as:

$$\alpha(X^*, X_t) = \min\left(\frac{f(X^*)q(X^* \mid X_t)}{f(X_t)q(X_t \mid X^*)}, 1\right)$$

So, more general, Metropolis-Hastings algorithm is:

1. Initialize chain with $X_0 : f(X_0) > 0$
2. Sample a point X^* from proposal distribution $q(X^* \mid X_t)$
3. Calculate $\alpha(X^*, X_t) = \min\left(\frac{f(X^*)q(X^* \mid X_{t-1})}{f(X_{t-1})q(X_{t-1} \mid X^*)}, 1\right)$
4. Except proposal point X^* with probability α.

The proposal distribution $q(\cdot \mid \cdot)$ can have any form and the stationary distribution of the chain will be $p(x)$. This can be seen from the following argument. The transition kernel form the Metropolis-Hastings algorithm is [40]

$$P(X_{t+1} \mid X_t) = q(X_{t+1} \mid X_t)\alpha(X_{t+1}, X_t) + I(X_{t+1} = X_t)\left(1 - \int q(Y \mid X_t)\alpha(Y, X_t)dy\right). \quad \textbf{(5.2)}$$

The first term in ((5.2) arises from acceptance of a candidate $Y = X_{t+1}$, and the second term arises from rejection, for all possible candidates Y. Using the fact that

$$p(X_t)q(X_{t+1} \mid X_t)\alpha(X_{t+1}, X_t) = p(X_{t+1})q(X_t \mid X_{t+1})\alpha(X_t, X_{t+1})$$

we obtain the *detailed balance equation:*

$$p(X_t)P(X_{t+1} \mid X_t) = p(X_{t+1})P(X_t \mid X_{t+1}). \quad \textbf{(5.3)}$$

Integrating both sides of ((5.3) with respect to X_t gives:

$$\int p(X_t)P(X_{t+1} \mid X_t)dX_t = p(X_{t+1}). \quad \textbf{(5.4)}$$

The left-hand side of equation ((5.4) gives the marginal distribution of X_{t+1} under the assumption that X_t is from $p(\cdot)$. There for ((5.4) says that if X_t is from $p(\cdot)$, then X_{t+1} will be also. Thus, once a sample from the stationary distribution has been obtained, all susequent samples wil be from that distribution.

Various methods can be derived from Metropolis-Hastings algorithm: independence sampler, single component Metropolis-Hastings, Gibs sampling, random-walk Metropolis, slice Gibs sampler. More information can be found in [24].

5.4 ADAPTIVE METROPOLIS ALGORITHM

Metropolis-Hastings algorithm and its modifications are useful when model has just few parameters and tuning of proposal distribution is relatively easy. However, as number of dimensions grows (e.g. in hierarchical modelling) satisfactory tuning is impossible.

Possible remedy is provided by adaptive algorithms, which use history of the process in order to "tune" the proposal distribution suitably. One of seminal papers is of Haario *et. al.*[19], where authors introduce adaptive Metropolis (AM) algorithm which adapts continuously to the target distribution. Significantly, the adaptation affects both the size and the spatial orientation of the proposal distribution. Moreover, this algorithm is straightforward to use and to implement in practice.

An important advantage of the AM algorithm is that it starts using the cumulating information right at the beginnings of the simulation. The rapid start of the adaptation ensures that the search becomes more effective at an early stage of the simulation, which diminishes the number of function evaluations needed.

Further will be sketched AM algorithm, as it is described in the original paper.

Assume that target distribution is supported on the subset $S \subset R^d$ and that it has the density $\pi(x)$ with respect to the Legesgue measure on S.

Suppose that at the moment $t-1$ we have sampled the states $X_0,...,X_{t-1}$. The proposal is as follows:

$$Y \sim q(\cdot \mid X_0,...,X_{t-1}) = Normal(X_{t-1}, C_t) \qquad \textbf{(5.5)}$$

The covariance matrix $C_t = \begin{cases} C_0, t \le t_0 \\ s_d \operatorname{cov}(X_0,...,X_{t-1}) + s_d \varepsilon I_d \end{cases}$, where ε is some small constant to ensure that covariance matrix is positive definite.

On the contrary to the Matropolis-Hastigs algorithm, AM generated chain is no longer Markovian, but ergodicity still holds.

6 SOFTWARE

The beginning of the 21st century found Bayesian statistics to be fashionable in science. But until late 1980s, Bayesian statistics were considered only as an interesting alternative to the „classical"theory. As history had proved, the main reason why Bayesian theory was unable to establish a foothold as a well-accepted quantitative approach for data analysis was the intractability involved in the calculation of the posterior distribution. Asymptotic methods had provided solutions to specific

problems, but no generalization was possible. Until the early 1990s two groups of statisticians had (re)discovered Markov Chain Monte Carlo (MCMC) methods. Physicists were familiar with MCMC methodology from the 1950s. Nick Metropolis and his associates had developed one of the first electronic supercomputers and had been testing their theories in physics using Monte Carlo techniques. Implementation of the MCMC methods in combination with the rapid evolution of personal computers made the new computational tool popular within a few years. Bayesian statistics suddenly became fashionable, opening new highways for statistical research. Using MCMC, we can now set up and estimate complicated models that describe and solve problems that could not be solved with traditional methods.

During 1990-1995, MCMC-related research focused on the implementation of new methods in various popular models. During the same period the early versions of BUGS software appeared. BUGS was computing-language-oriented software in which the user only needed to specify the structure of model. Then, BUGS was using MCMC methods to generate samples from the posterior distribution of the specified model.

The development of WinBUGS had proved valuable for the implementation of Bayesian models in a wide variety of scientific disciplines.

However, at some cases WinBUGS cannot be used for MCMC sampling since not all methods are implemented in it, e.g. adaptive Metropolis, which will be needed in our analysis. This calls for the use of another program to implement new MCMC methods, e.g. R software [32] which is open source and created mainly for statistical analysis purpose.

7 APPLICATION OF BAYESIAN FRAMEWORK FOR AGE-DEPENDENT RELIABILITY MODELLING

Ageing can be thought as age-dependent change of beliefs about systems parameters. Beliefs changes not just due to new data or other information (mentioned above) which becomes available in time, but also it changes due to flow of time.

One of the difficulties of Bayesian inference is inability to deal with changes of age-dependant parameter as a continuous process. This problem partially can be overcome by considering ageing (or degradation) as step-wise process, which is constant in some period of time and has value jump in other period. Mathematically this can be expressed as a jump process:

$$d(t) = \sum_{i=1}^{N-1} 1_{\{t_i < t < t_{i+1}\}} d(t_i), \qquad \textbf{7.1}$$

where $d(t)$ is any model of characteristic under consideration and constant $d(t_i)$ is value of characteristics at each time period t_i; N – number of time intervals.

Model of characteristic $d(t)$ can have any functional form. It can be linear, Weibull, or some other form. Depending on adopted formula, $d(t)$ will be based on vector of parameters $\Theta = \{\theta_1, ..., \theta_m\}$:

$$d(t) = d(t, \Theta). \quad \textbf{7.2}$$

If analysis considers more than one model, then indexation is used for different models, i.e. $d_i(t, \Theta_i)$, where d_i denotes ith model with Θ_i vector of parameters.

Modelling conception introduced above allows interpreting distribution of parameters as age-dependent. If prior knowledge and beliefs about systems parameters is represented by probability density distribution $\pi(\Theta)$ and statistical observations has likelihood $f(y \mid d(t))$, where $Y = (y_1, ..., y_N)$ is sample of observations, then, according to Bayes theorem, age-dependent beliefs about systems degradation or failure rate is expressed as posterior distribution:

$$\pi(\Theta \mid Y, t) = \frac{\pi(\Theta) f(Y \mid d(t, \Theta))}{\int_{\Omega} \pi(\Theta) f(Y \mid d(t, \Theta)) d\Theta}. \quad \textbf{7.3}$$

Assume that parameters $\theta_1, ..., \theta_m$ are a priori independent, then, according to definition of independent random variables, prior distribution of Θ can be expressed as:

$$\pi(\Theta) = \prod_{i=1}^{m} \pi_i(\theta_i), \quad \textbf{7.4}$$

where $\pi_i(\theta_i), i = \overline{1, m}$ are priors for components of vector Θ.

If data set contains n statistical observations, then posterior distribution is represented as:

$$\pi(\Theta \mid Y, t) = \frac{\prod_{i=1}^{m} \pi_i(\theta_i) \prod_{j=1}^{n} f(y_j \mid d(t_j, \Theta))}{\int_{\Omega} \prod_{i=1}^{m} \pi_i(\theta_i) \prod_{j=1}^{n} f(y_j \mid d(t_j, \Theta)) d\Theta}. \quad \textbf{7.5}$$

There are various techniques for model validation in Bayesian framework [24, 2, 11]. One of possible approaches to analyse model fitness is to use tail-area probability or as it is sometimes known, the posterior predictive p-value:

$$p = P(D(y^{rep}, \theta) > D(y, \theta) \mid y) = \int\int I_{[D(y^{rep}, \theta) > D(y, \theta)]} p(y^{rep} \mid \theta) p(\theta \mid y) dy^{rep} d\theta; \quad \textbf{7.6}$$

where y^{rep} is the replicated data that could have been observed, or, to think predictively, as the data that would appear if the experiment that produced y were replicated tomorrow with the same model [2]. Posterior p-value expresses the differences between statistical data and replicated. Rule of thumb is p-values close to 0.5 [24, 16].

$D(y,\theta)$ is discrepancy measure and can have any functional form, for example $D_1(y;\theta)=\sqrt{E(Y-E(Y))^2}$, $D_2(y;\theta)=\sum\frac{(y_t-E[y_t\mid\theta])^2}{Var[y_t\mid\theta]}$. Chi-square statistics $D_2(y;\theta)$ is quite popular among researchers, however as will be showed the use of just one discrepancy measure can be very misleading.

The use of discrepancy measures can be used to assess fitness of each model individually, i.e. rejection and acceptance of one model does not depend on other models.

Another possible way to analyse fitness of models is to use Deviance Information Criterion (DIC), which is already implemented in WinBUGS as inner function. DIC can be used to compare different models with each other. Spiegelhalter et al. [36] suggest the following rule of thumb: that models with DIC difference within the minimum value lower than two (2) deserve to be considered as equally well, while models with values ranging within 2-7 have considerably less support.

DIC of ith model is defined as:

$$DIC_i=-2\ln\left(L(\Theta\mid y,i)\right)+2p_D; \qquad \textbf{7.7}$$

where is the effective number of parameters [36].

For more information about Bayesian model selection can be found in [24, 11].

Usually it is the case when several trend models fits data almost equally well, i.e. possible set of "good" models can be represented as

$$\mathrm{M}=\left(d_1(t,\Theta_1),...,d_r(t,\Theta_r)\right), \qquad \textbf{7.8}$$

where $d_i(t,\Theta_i), i=\overline{1,r}$ are models which were considered as having good fit. In such circumstances uncertainties of modelling cannot be handled appropriately within classical statistical framework.

As noticed in [21], standard statistical practice ignores model uncertainty. Data analysts typically select a model from some class of models and then proceed as if the selected model had generated the data. This approach ignores the uncertainty in model selection, leading to over-confident inferences and decisions that are more risky that one thinks they are.

According to Hoeting [21], Bayesian averaging advantages include better average predictive performance than any single model that could be selected. Model averaging is more correct because it takes into account a source of uncertainty that analyses based on model selection ignore [8].

Denote $A(t)$ failure rate averaged over set of models M. Considering our notation, posterior probability of averaged age-dependent failure rate can be represented as:

$$p\left(A(t)\mid Y\right)=\sum_{j=1}^{r}p\left(A(t)\mid Y,d_j\left(t,\Theta_j\right)\right)p\left(d_j\left(t,\Theta_j\right)\mid Y\right), \qquad \textbf{(7.9)}$$

where $p\left(d_j\left(t,\Theta_j\right)|Y\right)$ is prior probability distribution of $d_j\left(t,\Theta_j\right)$ model, given set M of available models, $p\left(A(t)|Y,d_j\left(t,\Theta_j\right)\right)$ is a posterior distribution of quantity $A(t)$ under $d_j\left(t,\Theta_j\right)$. Posterior probability distribution for model M_j is given by

$$p\left(M_j|Y\right)=\frac{p\left(Y|d_j\left(t,\Theta_j\right)\right)p\left(d_j\left(t,\Theta_j\right)\right)}{\sum_{l=1}^{r}p\left(Y|d_l\left(t,\Theta_l\right)\right)p\left(d_l\left(t,\Theta_l\right)\right)}, \tag{7.10}$$

where $p\left(d_j\left(t,\Theta_j\right)\right)$ is prior probability distribution of models, $p\left(Y|d_j\left(t,\Theta_j\right)\right)$ is marginal likelihood conditional on model $d_j\left(t,\Theta_j\right)$.

In the case of non-informative prior distribution equal discrete probabilities can be assigned for each model $p\left(d_j\left(t,\Theta_j\right)\right)=\frac{1}{r}$ and posterior probability distribution for model $d_j\left(t,\Theta_j\right)$ becomes:

$$p\left(d_j\left(t,\Theta_j\right)|Y\right)=\frac{p\left(Y|d_j\left(t,\Theta_j\right)\right)\frac{1}{r}}{\sum_{l=1}^{r}p\left(Y|d_l\left(t,\Theta_l\right)\right)\frac{1}{r}}=\frac{p\left(Y|d_j\left(t,\Theta_j\right)\right)}{\sum_{l=1}^{r}p\left(Y|d_l\left(t,\Theta_l\right)\right)},\forall j=\overline{1,r}. \tag{7.11}$$

Even though Bayesian model averaging (BMA) seems to have advantages over one-model-fitting, little work has been done in the engineering field to address model uncertainties. Alvin et al. [4] used BMA to predict the vibration frequencies of a bracket component, Zhang and Mahadevan [45] applied it in fatigue reliability analysis on the butt welds of a steel bridge, and most recent work was done by Inseok Park et al. [22]. Authors analysed uncertainties of 4 finite elements models for laser peening process. However, all these works used relatively simple models and probabilistic approaches and there was no need to adopt advanced probability sampling techniques such as Markov Chain Monte Carlo methods [15].

8 BEHAVIOUR OF BAYESIAN ESTIMATES FOR SMALL DATA SAMPLES

It is well known that classical statistical methods fail when facing problems with small data samples. Certain parameters of models may not exist or may be on the boundary of the parameter space. Also maximum likelihood estimates are assumed to be approximately normally distributed and in small sample problems such assumptions can lead to very biased results.

From a Bayesian point of view, small sample problems can be dealt with without such shortfalls as in frequentist approach. However, such statement should be confirmed with empirical analysis.

In this chapter, behaviour of Bayesian methods in small sample problems will be investigated. Since in the case study Poisson data will be analysed, here I will narrow analysis with just Poisson distribution.

Suppose we have Poisson distribution with known rate curves. Trends with specific parameters are presented in (

Table 5.4.1). For each trend 1000 samples were generated to investigate bias of maximum likelihood estimator and Bayes estimator under quadratic loss, i.e. posterior mean estimator. Two sample sizes were used: 5 and 10 fake observations in each sample.

Table 5.4.1 Poisson intensity functions

	Poisson rate $\lambda(t)$
1.	$\exp(0.05t)$
2.	$\exp(0.5+0.05t)$
3.	$\exp(0.5+0.1t)$

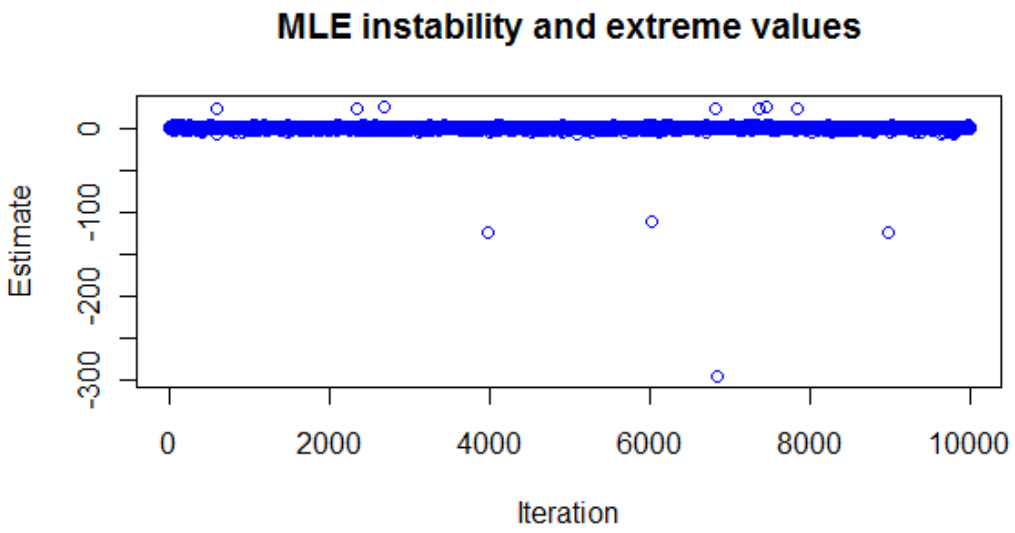


Figure 5.4.1 Illustration of MLE instability end extreme values of estimates

As for MLE, in unbounded parameter space estimates tend to take extreme values (Figure 5.4.1), and in bounded parameters space MLE often lie on the space boundaries (Figure 5.4.2).

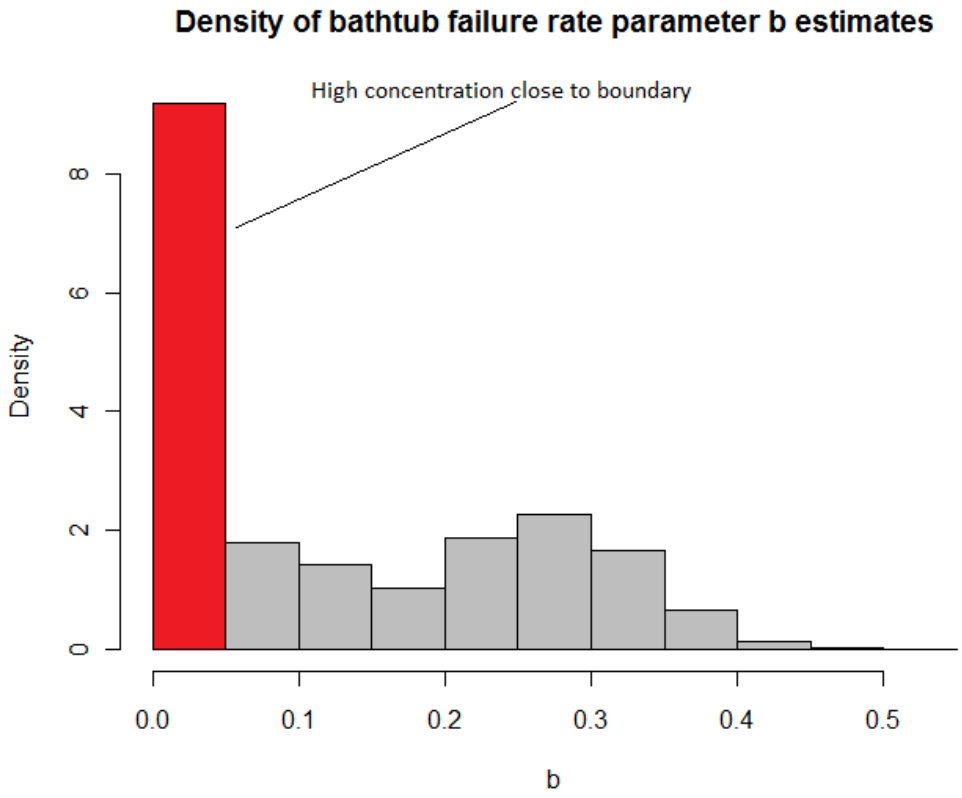


Figure 5.4.2 Distribution of MLE estimates of bathtub model parameter b

For Bayesian estimation, adaptive Metropolis method were used to assure good chain mixing, since in nonlinear regression parameters often are highly correlated.

Estimates by MLE and Byes estimator presented in Table 5.4.2.

Table 5.4.2 Bias from actual parameter value

Failure rate function		MLE		Bayes	
		N=5	N=10	N=5	N=10
1.	a	-0.596	-0.102	-0.331	-0.165
	b	0.035	0.052	0.050	0.052
2.	a	0.337	1.262	0.262	0.355
	b	0.052	0.056	0.048	0.054
3.	a	0.351	0.452	0.418	0.438
	b	0.112	0.102	0.103	0.099

Simulation results shows, that Bayes estimator outperforms MLE when sample size is 5 elements, but almost no difference can be observed for sample size of 10 elements. Although to more simulation analysis needed for more complex intensity curves (bathtub failure rate), but it can be conjectured that for such cases Bayes estimator would demonstrate better features compared to MLE, since as number of parameters grows, MLE requires more statistical data.

9 AGE-DEPENDENT PROBABILITIC ANALYSIS OF FAILURES IN GAS PIPELINES

Uncontrolled release of natural gas or loss of pressure in the system cause unsafe situations due to the potentially explosive mixture of gas and air [20]. Usual practice in assessment of such dangerous events is to consider failures rate of pipelines network as constant value. However, due to improvements in maintenance strategies, due to use of more advanced materials in the construction of new pipelines and in the repair of old ones, due to dynamic operating environment real failure rate is time-dependent.

If actual failure rate is higher than the value used in reliability and explosion assessment, then inferences made from such evaluation is overly optimistic and leads to underestimated risk. If actual failure rate is lower, then this leads to higher economical costs in risk management.

9.1 REVIEW OF STATISTICAL INFORMATION

UKOP (United Kingdom Onshore Pipeline Operators' Association) in its report for 2008th years [6] presents time-dependent (Figure 9.1.1) statistical estimates of failure rate, which were calculated every 5 years in 1969-2008 period.

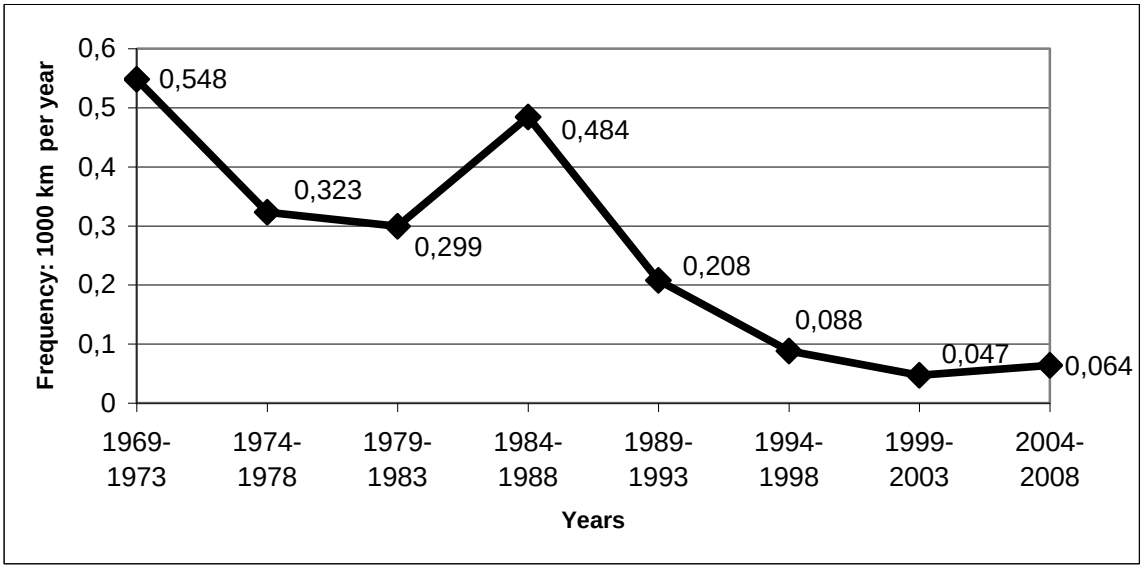


Figure 9.1.1 UKOP failure rate estimates of every 5 years

Failure[1] rate estimate of last 5 years (from 2004 till 2008) is 0.064 events for 1000 km per year, while general estimate of 1962-2008 period is 0.242 events for 1000 km per years. There is an increase over the last 5 year incident rate, but it is within the expected variation shown over the last ten years. An overview of the development of this failure frequency over the period 1962 to 2008 is shown in Figure 9.1.2.

[1] In this thesis, pipeline failure is defined as product (natural gas) loss incident, i.e. unintentional loss of product from the pipeline, excluding associated equipment (e.g. valves, compressors) or parts other than the pipeline itself

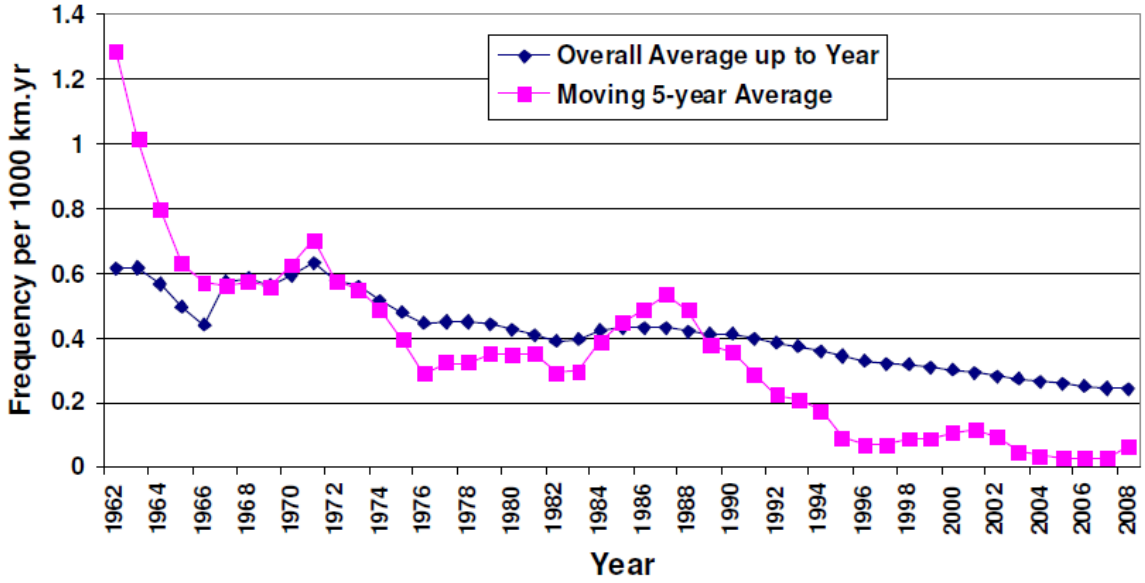


Figure 9.1.2 Development of overall incident frequency [6]

EGIG(European Gas Pipelines Incident Data Group) in its report [13] for period from 1970 till 2007 reports failure frequency equal to 3.7E-4. Overall length of pipeline network was 129719 km.

EGIG has investigated the relationship between the age of the pipelines and their failure frequencies to determine whether older pipelines fail, due to corrosion, more often than more recently constructed pipelines. The influence of the age of the pipelines on their failure frequencies has been studied in the ageing analysis presented in Figure 9.1.3.

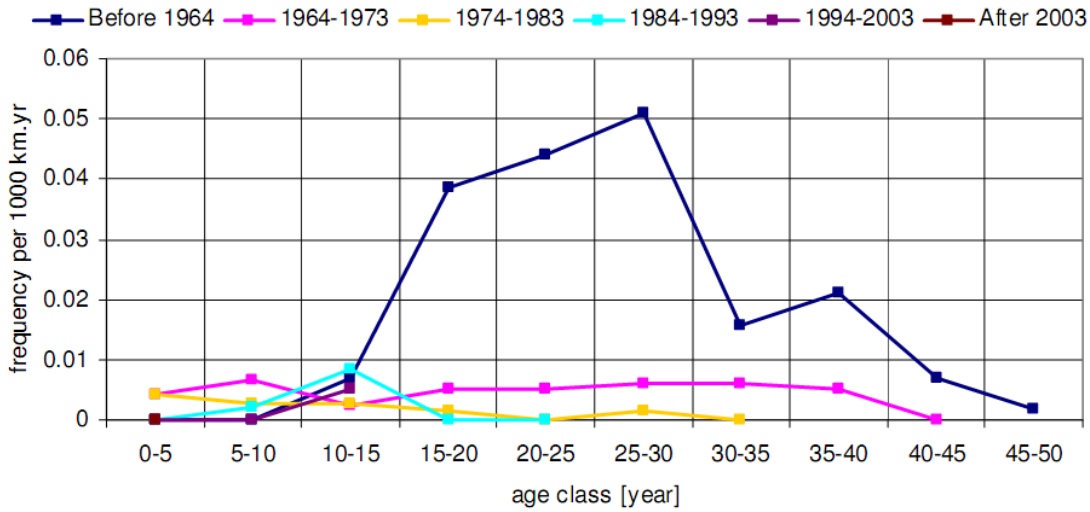


Figure 9.1.3 Ageing analysis (corrosion) relationship between age and construction year class

Early constructed pipelines had a higher failure frequency due to corrosion, in their early years, than recently constructed pipelines. In recent years, due to improved maintenance, pipelines age is no longer a major influence on the occurrence of corrosion failures.

Table 9.1.1 Distribution of failure frequencies

Category	Failure frequency (for 1 kilometer per 1 year)

Wall thickness less than 5 mm	4E-4
Wall thickness less than 10 mm	1.7E-4
Wall thickness less than 15 mm	8.1E-5
Wall thickness higher than 15 mm	4.1E-5

Distribution (Table 9.1.1) of failure frequencies according to pipe wall thickness is presented in report [12] for 2010th years of The International Association of Oil & Gas Producers. In 1172 events registered in EGIG database 11 ended with human injuries or death.

9.2 STOCHASTIC MODEL FOR FAILURE DATA

Suppose failure rate (Figure 9.1.1) follows decreasing trend function and data is generated by Gaussian nonlinear regression model with unknown dispersion, then full Bayesian model is as follow:

$$\begin{aligned} & Y_t \sim N\left(d\left(t,a,b,c\right),\sigma^2\right) \\ & d\left(t,a,b\right)=a+\frac{b}{1+t}+c^t, t=\overline{1,8}, \\ & a,b,c \sim uniform(0,K) \end{aligned} \qquad \textbf{(9.1)}$$

where K is some large constant (we used $K=1000$), ensuring that large enough space of states is explored by MCMC algorithm (Markov Chains Monte Carlo – is a family of specific algorithms, which allows generation of random number distributed by some distribution, known up to constant [18]). Uniform prior distributions for regression parameters are chose because no prior information is available to make any prior judgments about values of parameters.

MCMC simulation of stated model allows estimation of posterior distribution of regression parameters (Figure 9.2.1) and of dispersion parameter (Figure 9.2.2). These posterior distributions represent updated state of knowledge about variability of model parameters.

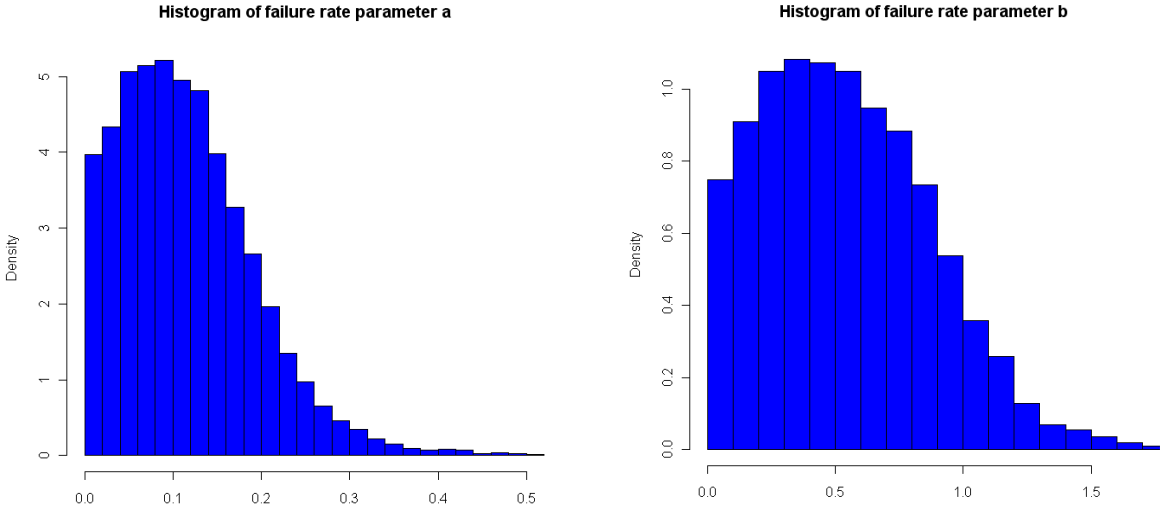


Figure 9.2.1 Posterior distributions of regression parameters a and b

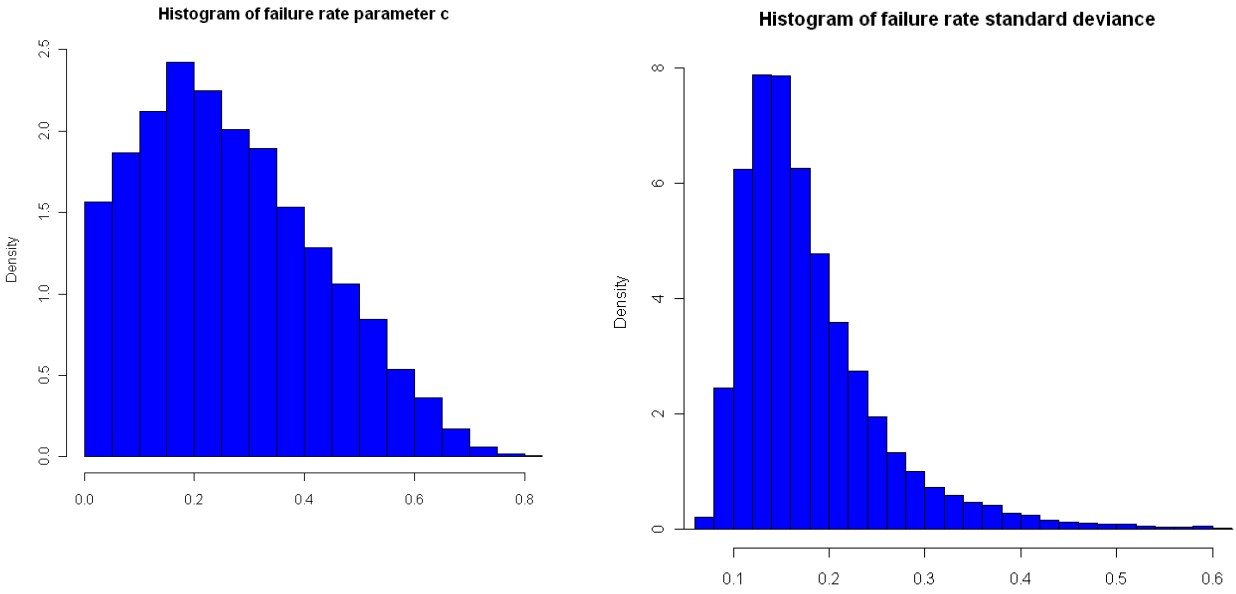


Figure 9.2.2 Posterior distributions of regression parameter c and model standard deviation

Point estimates and Bayesian confidence intervals, representing uncertainty about parameters after data were obtained are presented in Table 9.2.1.

Table 9.2.1 Bayesian point and interval estimates

Parameter	**Mean**	**95 % Confidence interval**
a	0.11	[0.0059;0.2549]
b	0.52	[0.0350;1.1682]
c	0.26	[0.0215;0.6112]
sigma	0.16	[0.0917; 0.2925]

It is worth to note, that Bayesian intervals have different meaning compared to frequentists confidence intervals: Bayesian intervals reflects probability of being in that interval while frequentists confidence intervals represents long-run frequency to „fall“ into calculated interval. For frequentists, at every new sample probability that parameter will be in previously computed confidence interval is either 1 or 0. Usually, which is a mistake, frequentists confidence intervals are interpreted as Bayesian confidence intervals.

Estimated failure rate of pipeline network under consideration is shown in Figure 9.2.3.

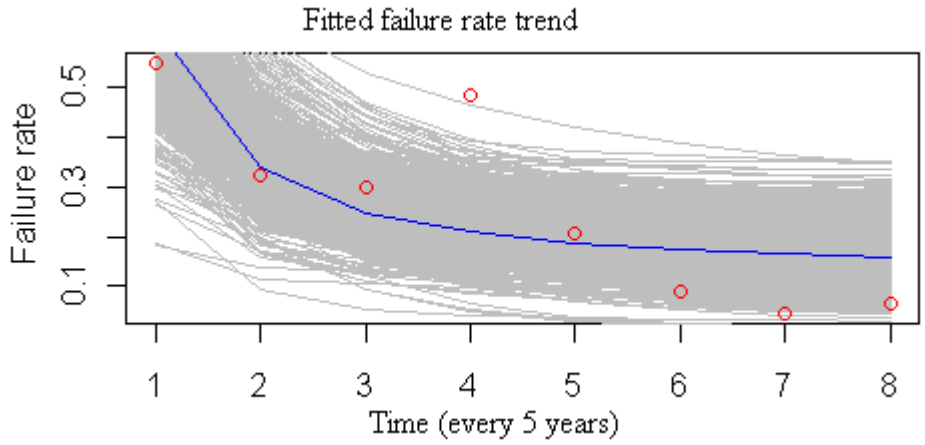


Figure 9.2.3 Age-dependent failure rate regression curve Bayesian and frequentists estimates

Failure rate with one term equal to constant a considered in this analysis were chose deliberately. Despite continuously improving maintenance strategies of natural gas transition network, it is unrealistic that failure rate will become equal to zero, so it is reasonable to analyze failure rate trend function with some limiting constant.

As is usually the case in reliability analysis, useful lifetime is of interest, i.e. system operation with constant failure rate. We will dismiss the case when ageing of system manifests, since it is highly unlikely, because of previously mentioned reasons for decreasing failure rate.

Further, in this section we will estimate time moment t^*, when failure rate approaches limiting constant (say, with error $\varepsilon = 0.01$) and whole lifetime of pipeline network can be divided into two sections: with decreasing and constant failure rate.

Time moment t^* is such that $\left|\lambda\left(t^*\right) - a\right| \le \varepsilon$ or $\left|\frac{b}{1+t^*} + c^{t^*}\right| \le 0.01$ and approximate solution is $t^* = 51$ time periods, which is equal to $5 \cdot 51 = 255$ years. These results show, that failure rate of gas grid settles down after quite long time and, since predictions for such time period would be very inaccurate, there is no reason to further analyze constant failure rate segment. Such segmentation would be useful in case when failure rate would approach constant value after relatively short time period (e.g. 10 or 20 years).

Failure rate estimate of UKOP natural gas transmission grid allows more advanced improvement of whole energy network reliability assessment and enables making more accurate predictions decisions. Well established practice to use constant failure rate for whole system lifetime when assessing reliability is harmful in terms of underestimated risk.

Age-dependent analysis presented in this section also allows modification of gas grid maintenance strategies dependent on network age, and, as a consequence, risk related to network accidents (such as gas leakage explosions) is more accurately assessed, as will be shown in next section, where age-dependent natural gas explosion probability will be estimated.

9.3 TIME-DEPENDENT EXPLOSION PROBABILITY OF NATURAL GAS PIPELINES NETWORK

The usual practice to calculate explosion probability near to nuclear power plant is to use constant pipeline failure rate [31]:

$$P = \lambda \cdot D \cdot f_s \cdot f_t \cdot f_d \cdot f_w \quad \textbf{(9.2)}$$

where λ is pipeline failure frequency, D is pipeline length, close to nuclear power plant, f_s is hazardous pipeline accidents frequency, f_t - frequency of accidents related to technical works performed close to site, f_d unnoticed and not repaired accidents, f_w - ratio of adverse weather conditions.

Further we will use estimates, presented in [13, 31]:

Table 9.3.1 Parameter values

Parameters	**Value**
λ	3.7E-4
f_s	0.32
f_t	0.25
f_d	0.1
f_w	0.5

Suppose, that $D = 1$ km, then probability of natural gas explosion near to nuclear power plant site is $P = 3.7 \cdot 10^{-4} \cdot 1 \cdot 0.32 \cdot 0.25 \cdot 0.1 \cdot 0.5 = 1.48 \cdot 10^{-6}$. However, the use of constant pipeline failure frequency can lead to overestimated or underestimated (depending whether actual failure rate is higher or lower than averaged constant value) gas explosion probability. So, to improve accuracy of explosion probability estimate and to better evaluate the risk, pipeline causes to nuclear power plant, time-dependent failure rate, like estimated in previous section, should be used.

Further we will use previously estimated failure rate $\lambda(t) = 0.11 + \frac{0.52}{1+t} + 0.26^t$. So, prediction of time-dependent probability of gas explosion close to (imaginary) nuclear power plant is as inFigure 9.3.1.

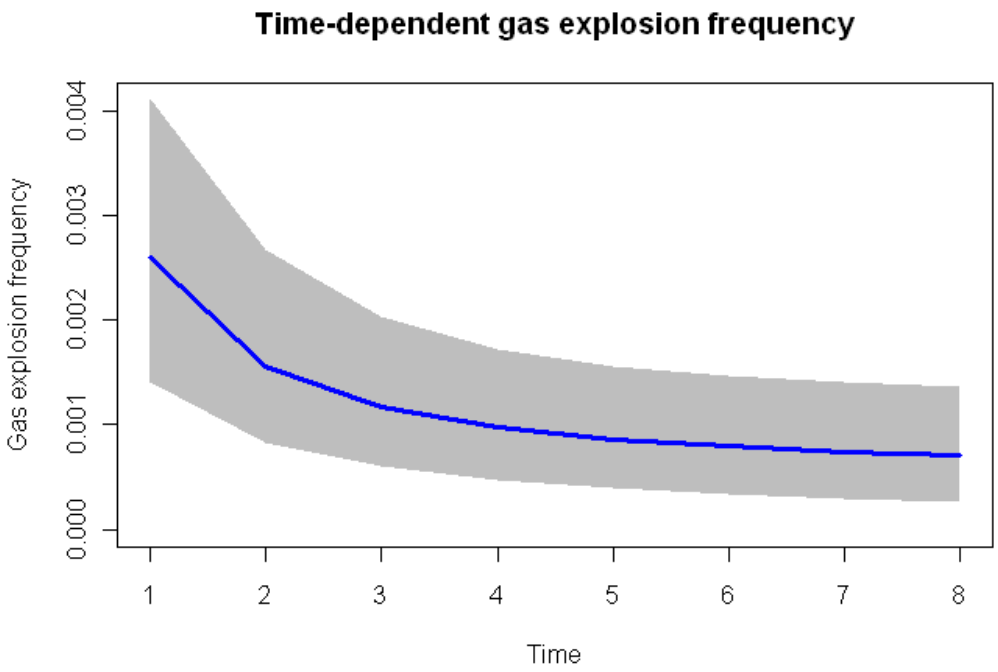


Figure 9.3.1 Time-dependent gas explosion probability near to nuclear power plant.

Gas explosion probability estimation approach is quite universal, because the object which is close to pipeline grid can be not necessarily nuclear power plant, it can be factory, houses and other important objects.

10 FAILURES OF ELECTRIC AND I&C COMPONENTS

10.1 DATA REPRESENTATION

Data set represents the failure and replacement dates of electrical instrumentation and control (I&C) components. The considered data is quite similar to the real (data were encoded and places where it was collected can't be identified) operating experience data collected in French or German nuclear power plants. In particular, it is a large sample that represents one technological group of continuously operating components. The data set contains records from type "T" reactors, which are operated by a single utility with a single management philosophy. The components are all of the same type (design, manufacturer, technology, etc.). The components operate in "A" environment having more stressful pressure and temperature. The scope of maintenance is the same for all components.

Data were collected in prior of eleven years, from January 1, 1990 through December 31, 2000. The components in the sample do not all have same date of being put into service, and as a consequence do not have the same ages at the beginning and end of observation. The failure counts were taken from a review of the maintenance data, so any reported date of failure is actually the date of the periodic test. A "critical" failure is one that causes the component to lose its safety function in a PSA model.

There are 20 units of type "T", each with 20 components of type A. The data collection period is eleven years, so there would be 4400 component-year except for the fact that some of the units were commissioned after the start of the data collection (Table 1).

Table 10.1.1 Failure data of I&C components under consideration

Unit	Start up	eqp	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	
T01	1988.01.06	20		0	0	0	0	0	3	0	0	1	0	0	1				5
T02	1989.01.01	20		1	1	0	0	0	1	0	0	1	0	0	0				4
T03	1987.01.04	20			0	0	0	1	1	0	0	0	0	0	0	0			2
T04	1988.01.02	20		0	0	0	0	0	0	0	1	0	0	0	0				1
T05	1991.01.02	20	0	0	0	1	0	0	0	0	1	0							2
T06	1992.01.01	20	0	0	0	0	0	0	0	0	0	0							0
T07	1986.01.12	20				0	4	0	1	1	0	0	4	0	1	0	2		13
T08	1987.01.03	20			0	0	0	0	1	3	2	0	1	1	1	1			10
T09	1991.01.02	20	0	0	2	0	0	0	0	0	0	0							2
T10	1994.01.03	20	0	0	0	0	0	0	1										1
T11	1988.01.02	20		0	0	0	0	0	0	0	0	0	0	0	1				1
T12	1989.01.05	20	0	0	0	0	0	0	0	0	0	0	0	0	0				0
T13	1985.01.12	20					2	0	0	1	2	1	4	2	3	3	2	0	20
T14	1985.01.12	20					0	1	1	2	2	2	6	3	0	3	0	0	20
T15	1986.01.02	20				0	0	1	2	3	1	2	0	0	1	1	5		16
T16	1986.01.06	20				0	2	1	2	0	0	0	0	1	0	5	1		12
T17	1990.01.12	20	0	0	0	0	2	2	0	0	0	0	0						4
T18	1992.01.11	20	1	0	0	0	0	1	2	0	0								4
T19	1986.01.05	20				0	0	1	1	0	3	0	1	1	0	1	5		13
T20	1987.01.03	20			0	0	0	0	0	1	0	1	0	1	2	2			7
Number of failures			1	1	3	1	10	8	16	11	12	8	16	9	10	16	15	0	**137**
Operating time			126,56	171,62	231,36	314,8	396,6	400	396,76	380	363,34	336,73	281,68	273,42	288,44	168,58	85,16	3,36	**4218,41**
λ			0,0079	0,0058	0,013	0,0032	0,0252	0,02	0,0403	0,0289	0,033	0,0238	0,0568	0,0329	0,0347	0,0949	0,1761	0	**0,032477**

Simple failure rate (Figure 10.1.1) and cumulative failure rate (Figure 10.1.2) plot gives first impression about failure behaviour over time: failure rate increases in time showing equipment degradation due to ageing.

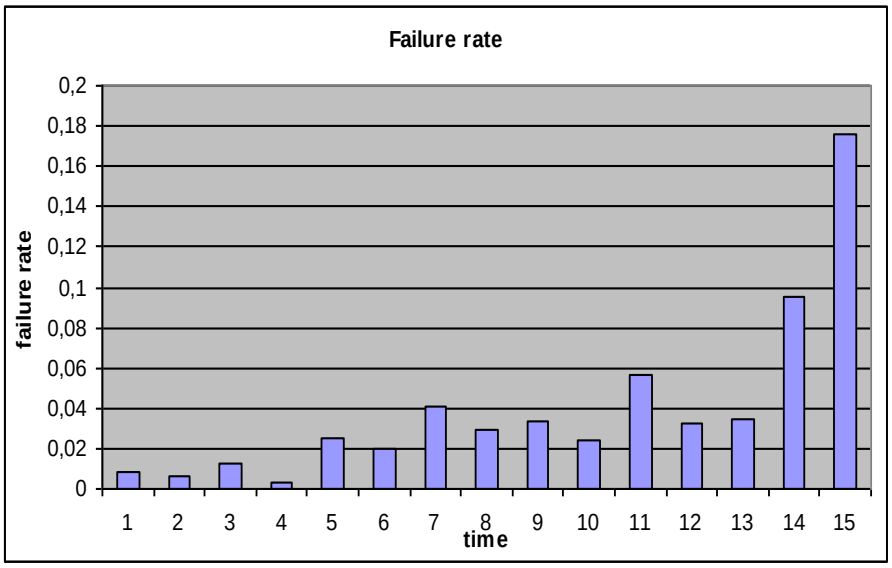


Figure 10.1.1 Failure rate

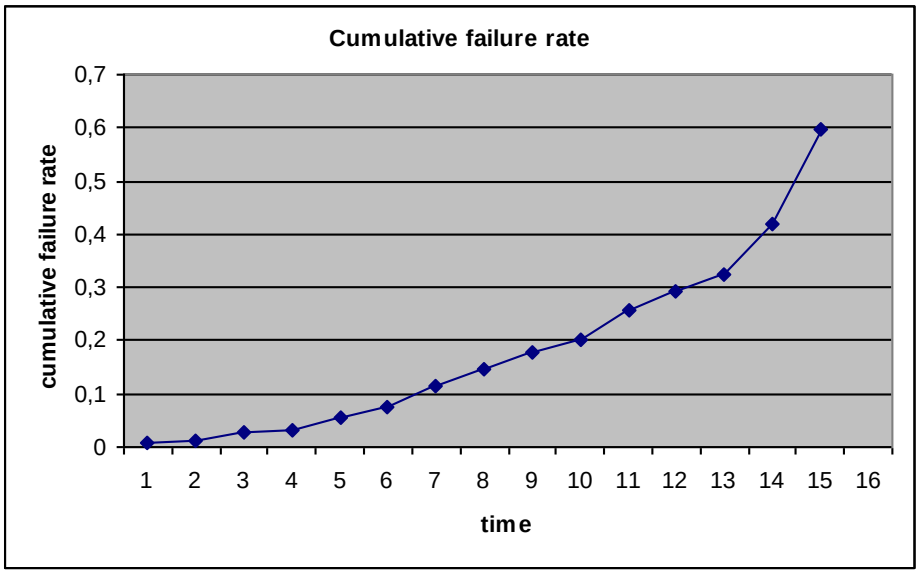


Figure 10.1.2 Cumulative failure rate

In report of JRC Institute for Energy [9] can be found full frequentist analysis of data in Table 4.2.1. In that analysis side-by-side 90% confidence intervals are calculated and plotted, nonparametric and parametric test are performed to validate ageing in observed data.

10.2 BAYESIAN MODEL FOR PIECEWISE HOMOGENEOUS POISSON COUNT DATA

In this analysis, failure rates are considered as constant values in each year, but at every year this value jumps at the value which can be calculated from linear, Weibull or other model.

Consider as the model for the failure rate $\{\lambda(t); t \geq 0\}$ a jump process structure described above:

$$\lambda(t) = \sum_{i=1}^{N} 1_{\{t_i < t \leq t_{i+1}\}} \lambda_i \,. \qquad \textbf{(10.1)}$$

In each year period failures occurs as Poisson process but with different rate parameter $\lambda_i, i = 1,2,...,15$. In every time period (which in this case is equal to one year) equipment was in operation for τ_i time (operating time). Denote number of failure that occurred in one year as N_i. Probability of failure can be expressed as:

$$P(N_i = k) = \frac{e^{-\lambda_i \tau_i} (\tau_i \lambda_i)^k}{k!}. \qquad \textbf{(10.2)}$$

Likelihood function, that contains all information obtained from data, is:

$$L(\Theta) = P(y \mid \Theta) = \prod_{i=1}^{n} \exp\{-\lambda(t_i, \Theta)\tau_i\} \frac{(\lambda(t_i, \Theta)\tau_i)^{N_i}}{N_i!}. \qquad \textbf{(10.3)}$$

Since there is no available information in data source [9] about which particular I&C components were under observation, prior distribution for parameters of failure trend function is chosen as diffuse distribution. In WinBUGS implementation diffuse prior gamma distribution were assigned for all parameters and in all models except for Xie and Lai model – for one parameter beta distribution where assigned. So for Xie and Lai failure rate trend model joint prior distribution can be expressed as

$$\pi(\Theta) = \prod_{i=1}^{3} \frac{b^a \theta_i^{a-1} e^{-b\theta_i}}{\Gamma(a)} \cdot \frac{1}{B(\alpha, \beta)} \theta_4^{\alpha-1} (1-\theta_4)^{\beta-1}. \qquad \textbf{(10.4)}$$

When for other models join prior distribution is:

$$\pi(\Theta) = \prod_{i=1}^{m} \frac{b^a \theta_i^{a-1} e^{-b\theta_i}}{\Gamma(a)}. \qquad \textbf{(10.5)}$$

In further analysis we considered 5 failure rate trend models described previously. Linear, exponential and power models represent class of trends which is common in ageing analysis and Makeham and Xie & Lai models represents more flexible bathtub trend class. We excluded constant failure rate model because ageing has been already validated in [9].

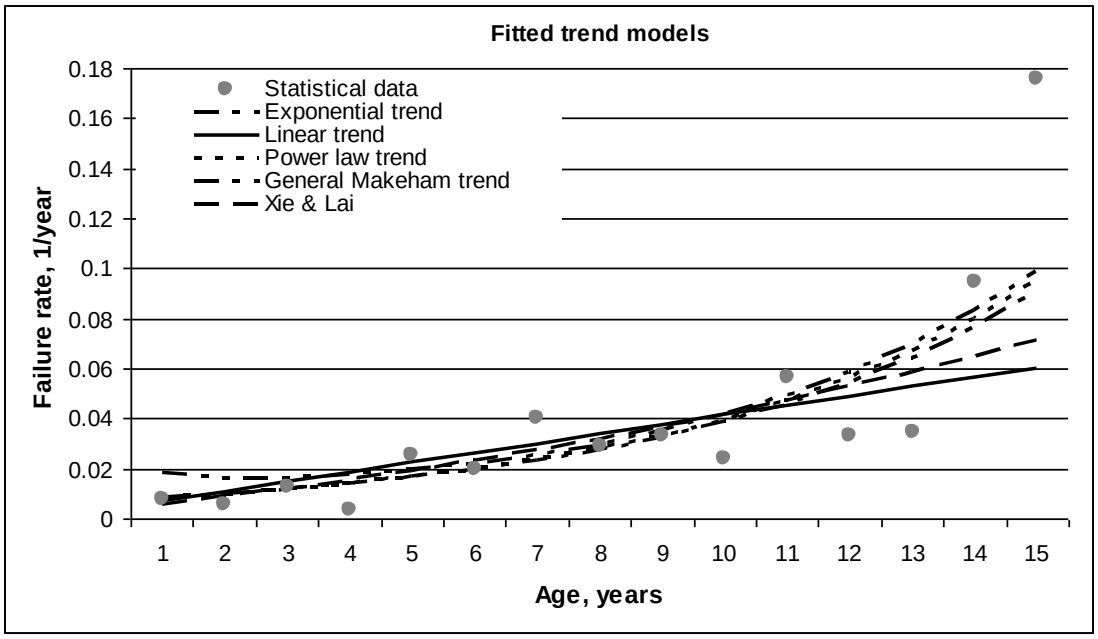


Figure 10.2.1 Comparative representations of fitted trend models

10.3 MODEL FITTING AND SCREENING

Adaptive Metropolis algorithm presented in theoretical part of this thesis, proves to be very useful in nonlinear regression modelling: convergence of generated chains are achieved without problems. However, in Xie and Lai model algorithm had to run a bit longer to achieve good chain mixing and convergence (Figure 10.3.1).

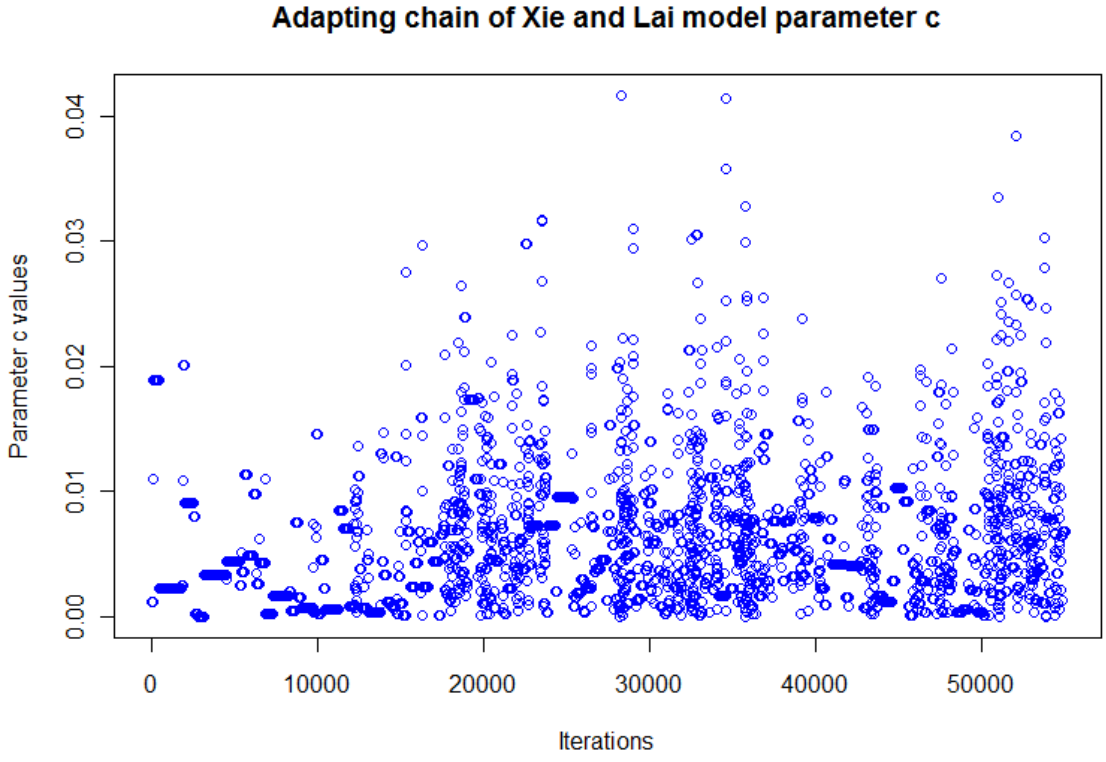


Figure 10.3.1 Nonhomogeneous Markov chain of Xie & Lai parameter c

To ensure, that the chain is converged to stationary distribution, cumulative means of parameters were monitored.

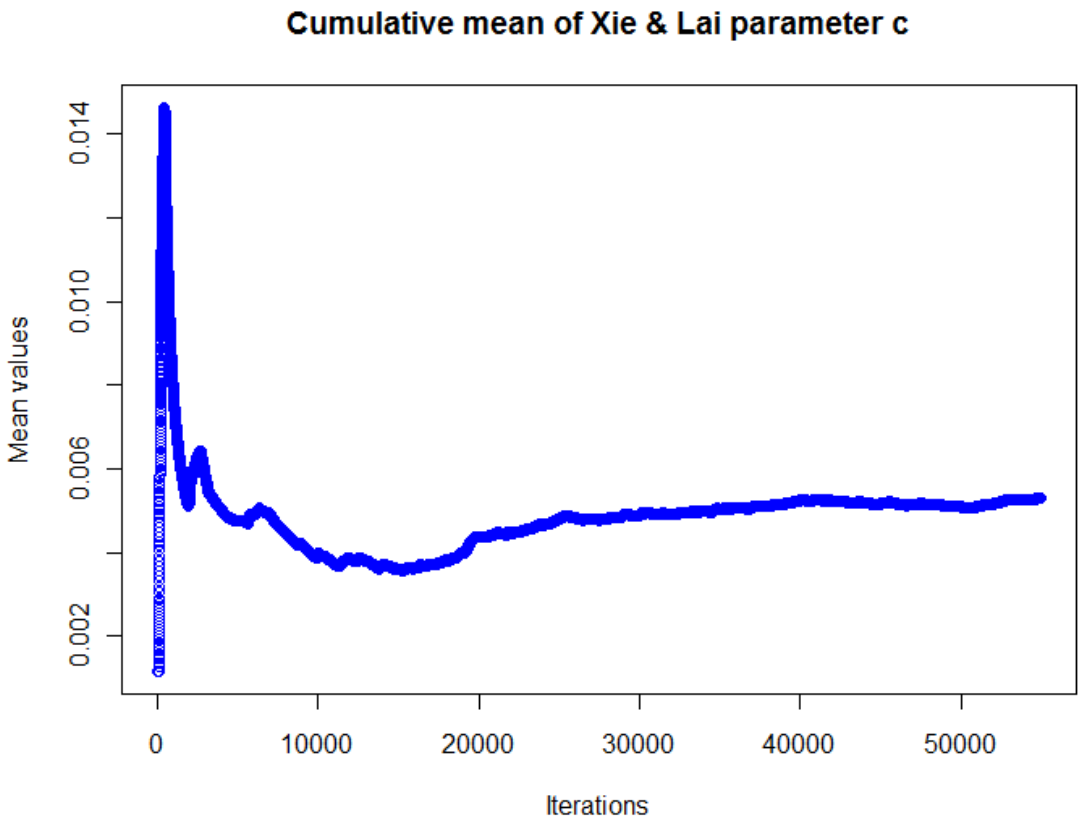


Figure 10.3.2 Cumulative mean of Xie and Lai model parameter c

Posterior joint marginal distributions for Xie and Lai trend model parameters:

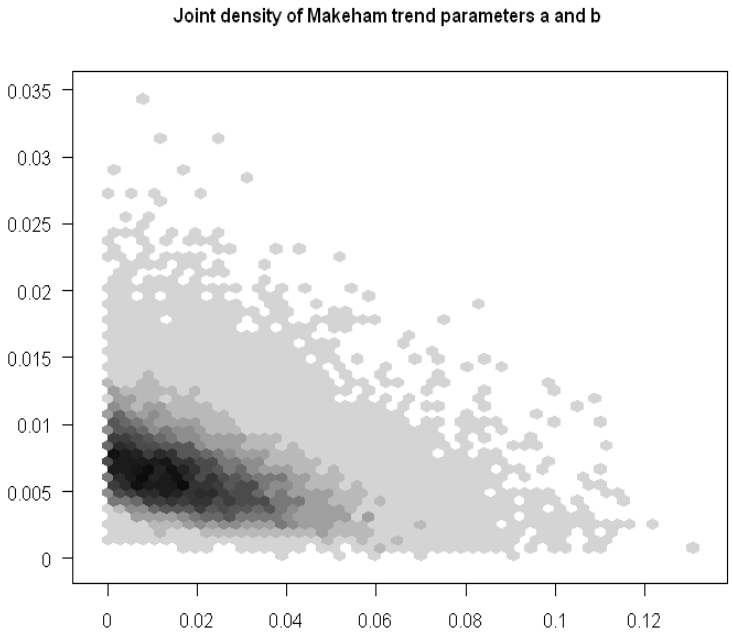


Figure 10.3.3 Joint marginal density of Makeham model parameters a and b

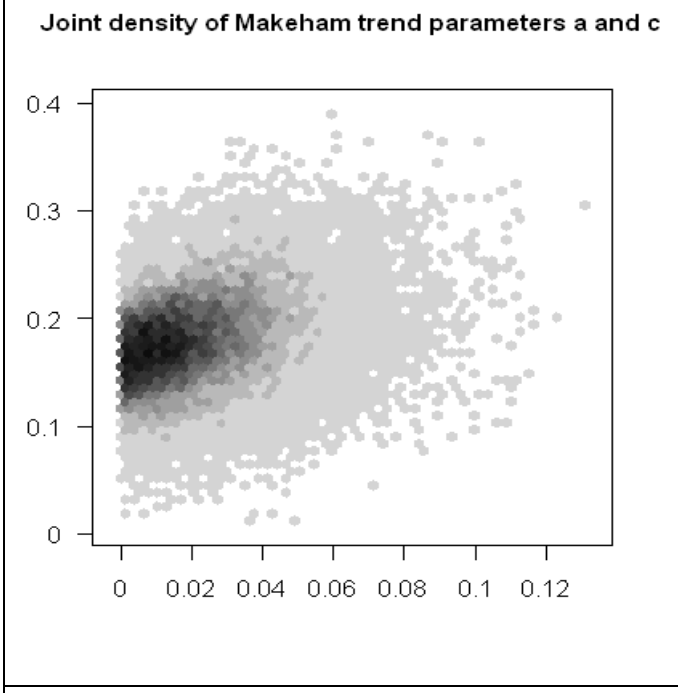


Figure 10.3.4 Joint marginal density of Makeham model parameters a and c

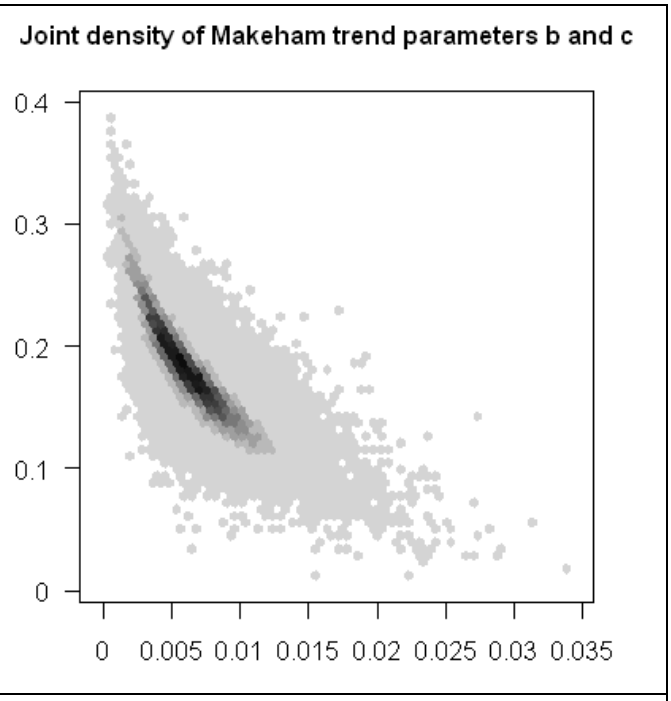


Figure 10.3.5 Joint marginal density of Makeham model parameters a and c

Age-dependent evolution of generalized Makeham failure rate is captured in (Figure 10.3.6). It is one of Bayesian advantages to be able to naturally quantify uncertainties present in inferences.

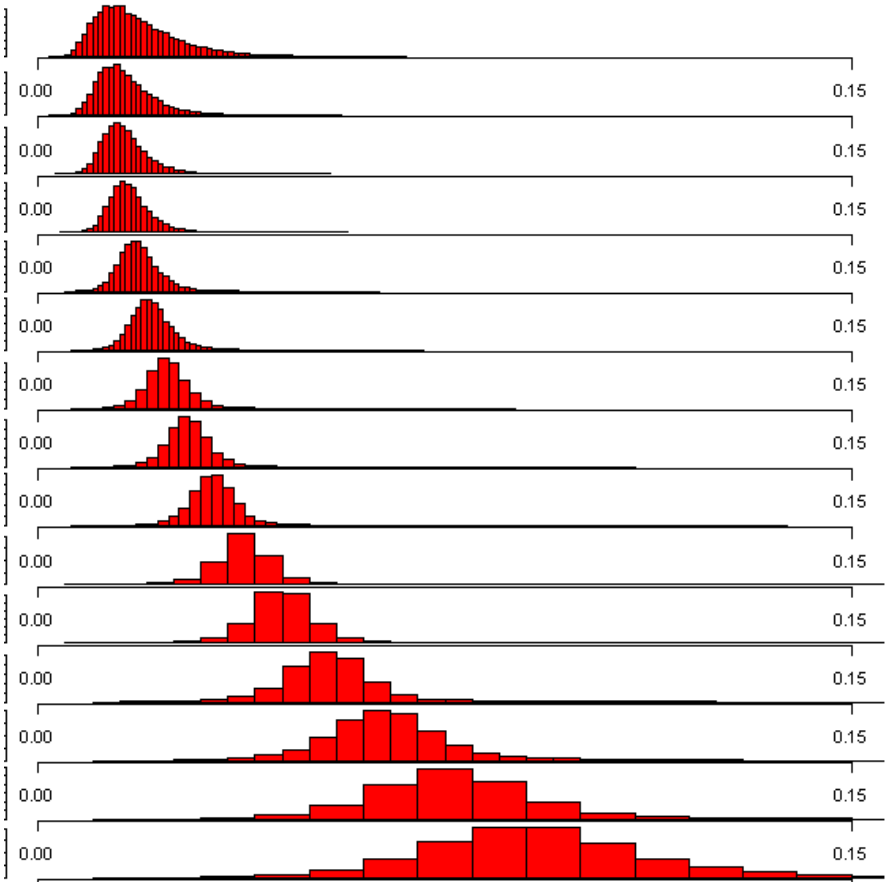


Figure 10.3.6 Age-dependent evolution of uncertainties of generalized Makeham failure rate

Estimated posterior p-values for different failure rate models are in Table 10.3.1:

Table 10.3.1 Posterior p-values for different failure rate models

	Linear	Exponential	Power	Makeham	Xie & Lai
p_1	0.5458	0.6333	0.7134	0.6178	0.7006
p_2	0.0042	0.0278	0.0102	0.0306	0.011

As can be seen from posterior p-values p_2 presented in Table 1, none of proposed failure rate trend models gives good enough fit and all models should be rejected. However, p-values p_1 shows satisfactory discrimination abilities – linear and generalized Makeham trend models can be interpreted as better fit than exponential and power low failure rate trend models.

It is worth to take notice of chi-square discrepancy measures inability to assess model goodness-of-fit, even though graphical investigation shows quite tolerable fitness. Authors think that, one of possible reasons is because of heteroscedasticity in errors (Figure 10.3.7).

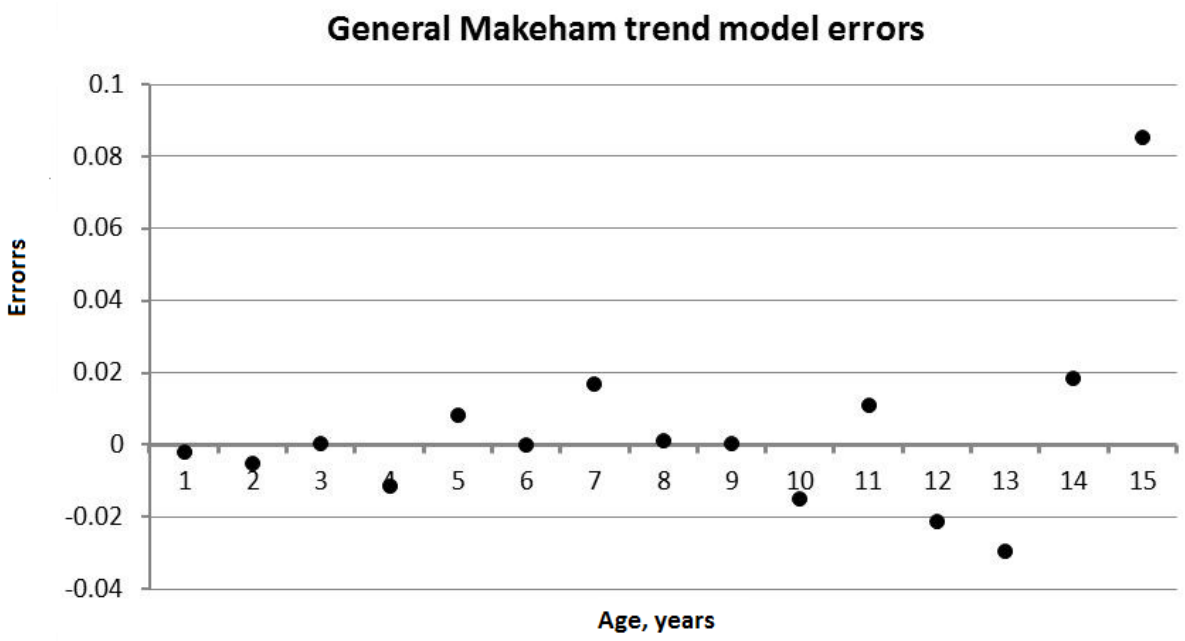


Figure 10.3.7 Heteroscedasticity of nonlinear generalized Makeham model errors

Analysis of fake data showed that chi-square discrepancy measure behaves well when data is transformed to be linear, in this case errors heteroscedasticity is not so obvious and does not cause any trouble (if heteroscedasticity is the main problem at all, because this is just hypothesis of authors and needs separate investigation). However, data transformation into linear form is not the best solution, because it highly depends on the problem at hand what kind of transformation will result into desirable linearity. Also any transformation, either of data or of parameters in model, leads to different results, since invariance under transformation is not preserved, e.g. deviance information criterion values are dependent on which transformation is applied.

Even though standard deviation measure seems to work, but it might be that applied to another data sample it fails as is the case with chi-square measure in this problem. This leads to the conclusion that discrepancy measures (and as a consequence, posterior predictive p values) does not provide automatic model assessment tool for practitioners.

It is well known that more complex curves will fit data more precisely, but fitness of very complex models can lead to over fitting (e.g. perfect fitness can be achieved by splines, but this apparently leads to nonsensical inference).

Nevertheless, this obscurity can be solved by using DIC measure. This criterion naturally adopts Occam's razor principle, because it incorporates penalty - the effective number of parameters: more complex models will be penalized more severely. DIC values for all models under consideration are presented in Table 10.3.2.

Table 10.3.2 Values of Deviance Information Criterion

Model	Linear	Exponential	Power law	Generalized Makeham	Xie & Lai
DIC	91.39	86.48	86.42	94	88

As can be seen from DIC values, exponential and power law model shows best fit. Also, Xie & Lai model can be accepted.

Two measures of fitness – discrepancy measure and DIC – shows different results and unambiguous answer cannot be given. Preference to one model over another can lead to too pessimistic or optimistic predictions of ageing phenomena behaviour. Such uncertainty related to the selection of model for further use has to be quantified to make sure that applications of model will not be influenced on incorrect choice of trend. Such quantification will be demonstrated in further analysis where Bayesian model averaging (BMA) will be applied.

10.4 BAYESIAN POSTERIOR MODEL AVERAGING

As was concluded previously, discrepancy measure and DIC gave quite ambiguous results; subsets of models, selected by these criterions are not exactly the same. In practice, usual decision is to adopt just one model, but as mentioned in theoretical part, this could lead to overoptimistic results if model uncertainty is not incorporated into modelling process.

In this part of paper application of Bayesian model averaging to analyse age-dependent failures will be demonstrated. We will perform averaging procedure for all models that were considered in this paper. To be able to average over set M of models, probabilities of each model has to be obtained by calculating marginal likelihoods. To do so, power posteriors were used [14]. Obtained probabilities are presented in Table 10.4.1.

Table 10.4.1 List of probabilities of analysed models

	Linear	Exponential	Power	Gen. Makeham	Xie & Lai
$p\left(d_j\left(t,\Theta_j\right)\mid Y\right)$	0.027	0.456	0.444	0.046	0.026

Calculated probabilities partially justifies assessments made by DIC and do not confirm conclusions based on discrepancy measures. Posterior uncertainty limits for both exponential and power trends are presented in Figure 10.4.1.

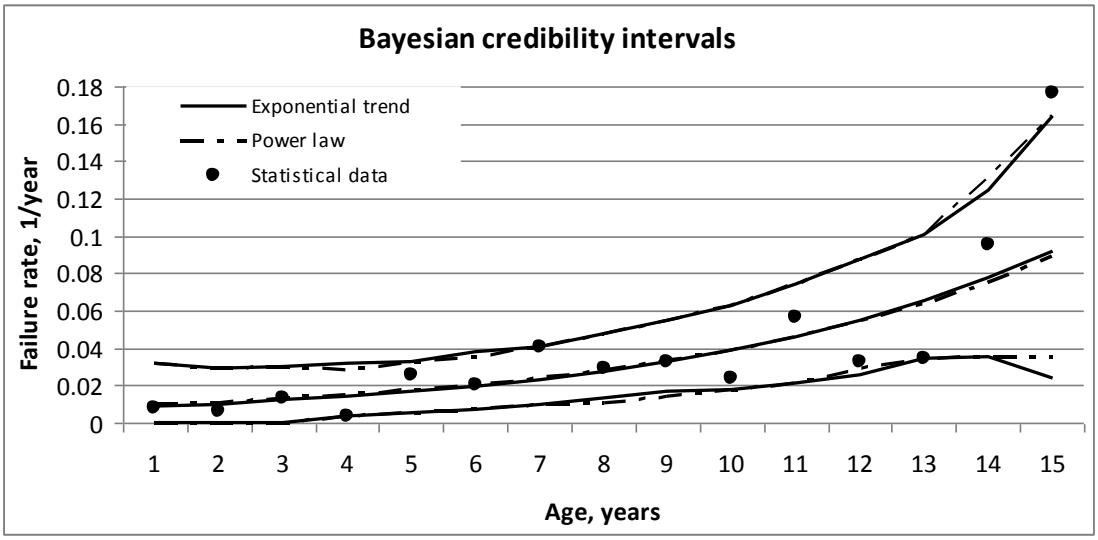


Figure 10.4.1 Uncertainty intervals for exponential and power law posterior failures

Authors think, that Bayesian averaging procedure can be good alternative to various goodness-of-fit approaches since it prevents decision maker of exclusion of models which have good fit and could lead to reasonable posterior inferences.

The results of Bayesian model averaging are presented in Table 10.4.2.

Table 10.4.2 Posterior averaged values of failure rate together with uncertainty limits

Statistical data		Averaged	Upper credibility interval	Lower credibility interval
Age	Failure rate	failure rate	bound	bound
1	0.007901	0.00639	0.000364	0.0167
2	0.005827	0.00877	0.000991	0.01941
3	0.012967	0.01151	0.003294	0.02178
4	0.003177	0.01465	0.006547	0.02454
5	0.025214	0.01799	0.009665	0.02769
6	0.020000	0.02166	0.01253	0.03218
7	0.040327	0.02568	0.01564	0.03691
8	0.028947	0.03013	0.01908	0.04246
9	0.033027	0.03513	0.0229	0.04869
10	0.023758	0.04059	0.02697	0.05571
11	0.056802	0.04672	0.03053	0.06469
12	0.032916	0.05357	0.03579	0.0734
13	0.034669	0.06136	0.04211	0.08302
14	0.094910	0.07014	0.04368	0.09992
15	0.176100	0.08016	0.0426	0.125

Posterior averaged failure rate values are better calibrated, because not only uncertainty in parameters accounted, but also uncertainty regarding model selection. This averaging procedure prevented us from exclusion of models by giving very small weights for those which were unlikely, according to the evidence contained in marginal likelihood. One drawback on Bayesian posterior averaging is that if we construct set of models that all have very poor fit then averaged posterior quantities will also fit inadequately. In this case DIC could be applied to eliminate models with highest deviance information criterion values. Also, accounting of model uncertainty cannot be fully performed, because there is infinite number of possible models.

11 CONCLUSIONS AND FINAL REMARKS

- In this thesis I presented general methodology of Bayesian methods application for age-dependant analysis. It was showed that this methodology is able to deal with disperse and small data amount along with multiple parameter set (Makeham and Xie & Lai trend models);
- Observed instability of MLE for small samples and when parameter space were bounded;
- Metropolis-Hastings method was compared to adaptive Metropolis method and results showed that the latter is able to deal with nonlinear modelling and highly correlated parameters;
- Proposed methodology was successfully applied for ageing analysis of electrical I&C components. This application was carried in terms of piecewise homogeneous Poisson model with several failure trends;
- For fitting and screening of various trend models, it was noticed that none of model selection approaches can give unambiguous answer. P-values can be quite misleading and can either show no discriminatory abilities (as in case of chi-square p-value) or can suggest more than one model as having good fit (as in case of standard deviation p-value). It was concluded that failure of p-values, based on chi-square discrepancy measure, could be due to heteroscedasticity present in model errors;
- Performed posterior Bayesian averaging procedure over set of selected trends resulted to better predictive performance, because averaged future failure rates will not be underestimated in terms of their uncertainties;
- This paper and its results can be used as groundwork for further assessment of ageing systems, structures and components. Its generality and idea, that ageing or degradation can be thought as age-dependent change of beliefs about system reliability parameters, allows analysis of wide spectrum of problems - it can be stochastic behaviour of crack growth (in this case characteristic $d(t)$ of interest would be crack growth rate), it can be degradation modelling as transitions through Markovian states ($d(t)$ could be transition rates, time-homogeneous or time-inhomogeneous, between degradation states), etc;
- Up-to-date statistical natural gas pipeline grid data were presented and used to estimate time-dependent failure rate and gas explosion probability which led to more accurate failure rate assessment;
- Bayesian methods allowed more robust estimation of time-dependent failure rate parameters; furthermore, uncertainties of these parameters were also obtained and used to

estimate confidence intervals which are more easily interpretable (compared to frequentists);

- Estimated time point when failure rate decreases to constant value (with some error ε) showed that there is no necessity to divide failure rate into two segments: strictly decreasing and constant;
- Time-dependent failure rate is advantageous for development of risk-informed maintenance strategies of pipeline grid, also for evaluating risk at different network points – this can be done by using time-dependent gas explosion probabilities instead of constant.

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12 PUBLICATIONS AND CONFERENCES

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13 APPENDIX

13.1 POSTERIOR QUANTITIES OF INTEREST

1. Power law model: $\lambda(t) = ab^t$;

Posterior joint distribution of parameters a and b:

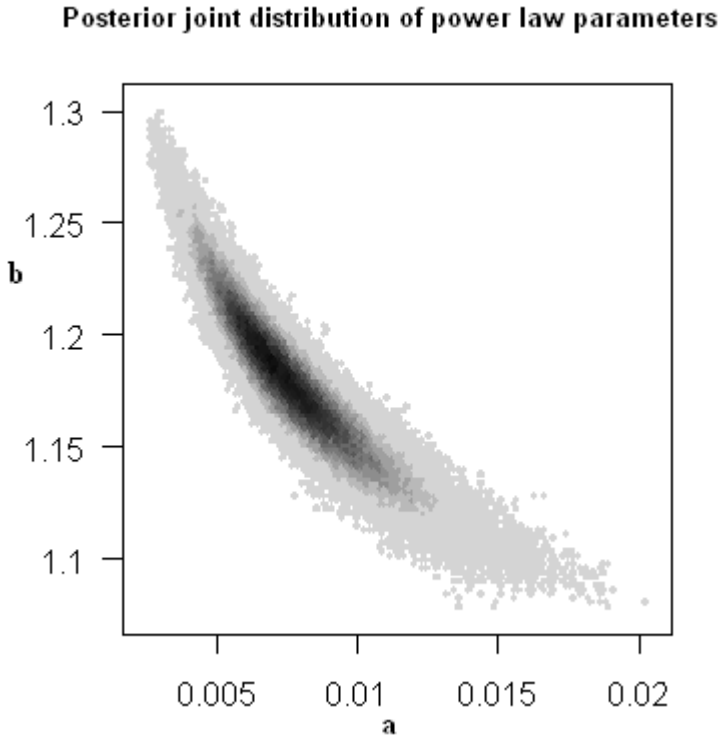

Figure 13.1.1 Joint distribution of power law model parameters

Posterior marginal distribution of model parameters:

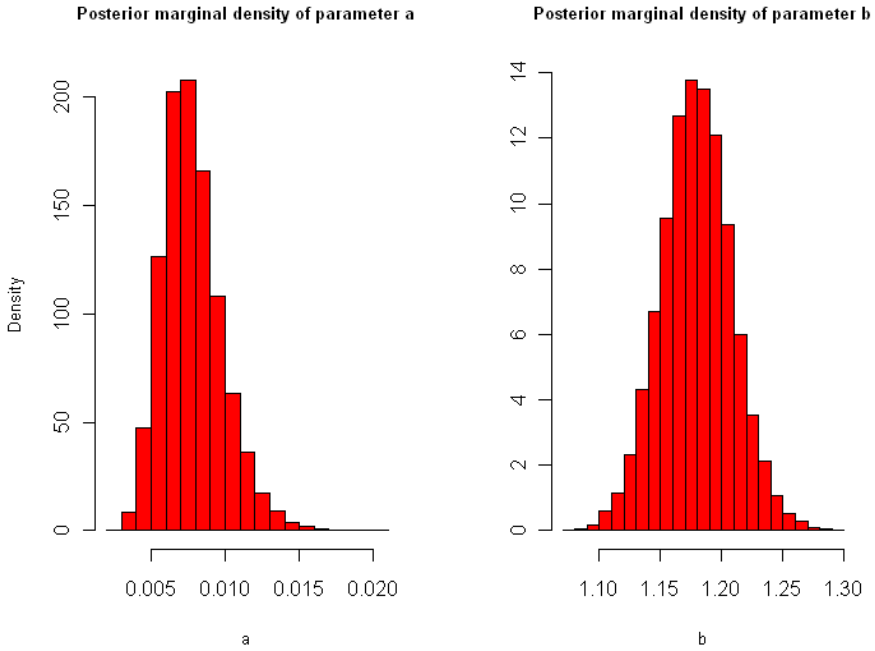

Figure 13.1.2 Posterior marginal density estimates of power law parameters

Age-dependent evolution of uncertainties, present in failure rate estimates

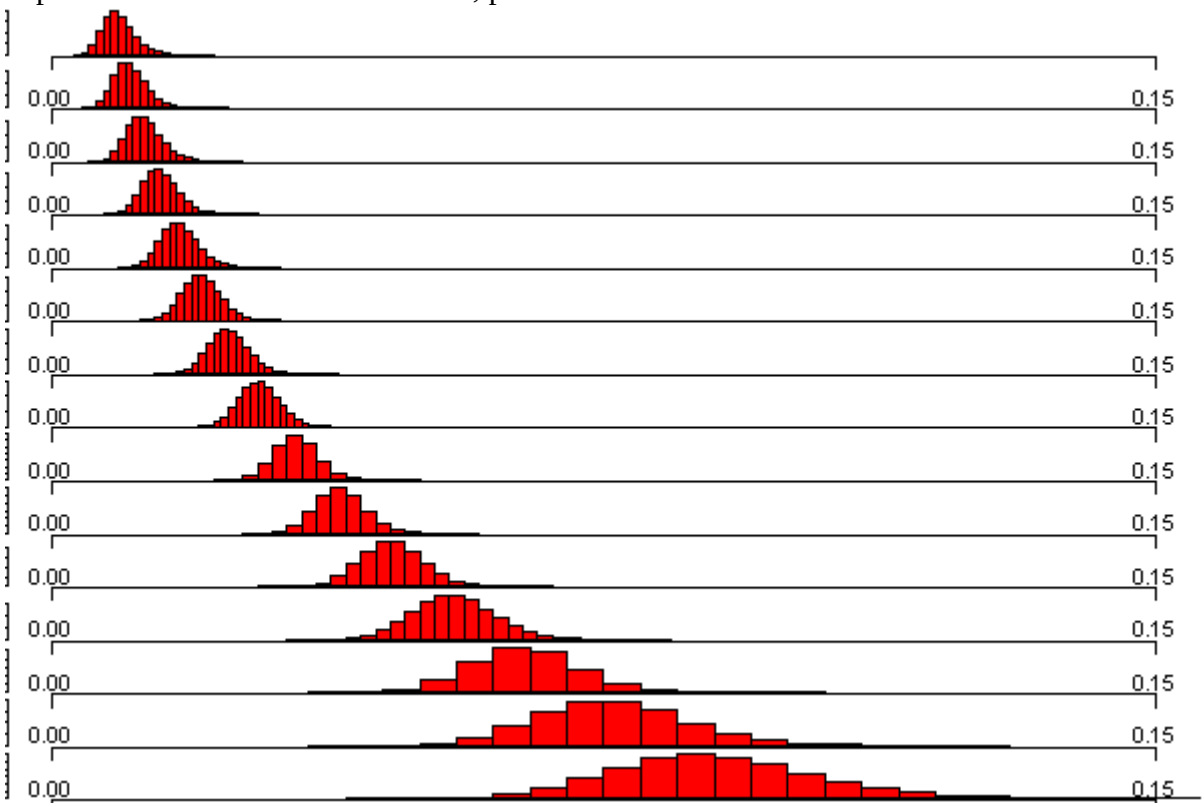

Figure 13.1.3 Age-dependent evolution of uncertainties of power law failure trend estimates

2. Linear model: $\lambda(t) = a + bt$;

Posterior joint distribution of parameters a and b:

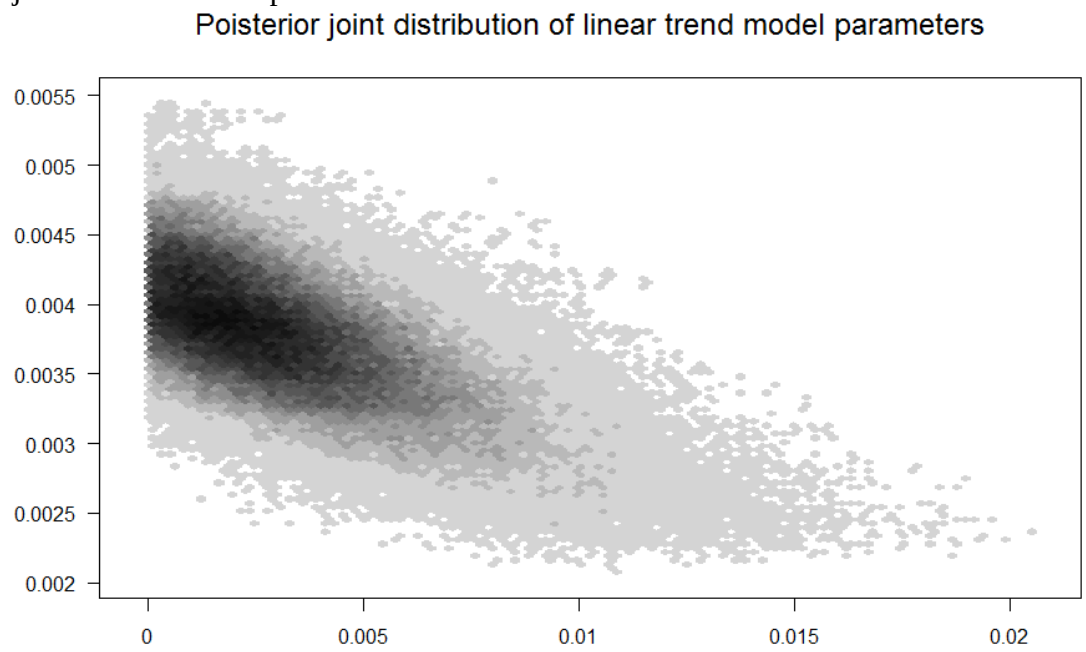

Figure 13.1.4 Joint distribution of linear model parameters

Posterior marginal distribution of model parameters:

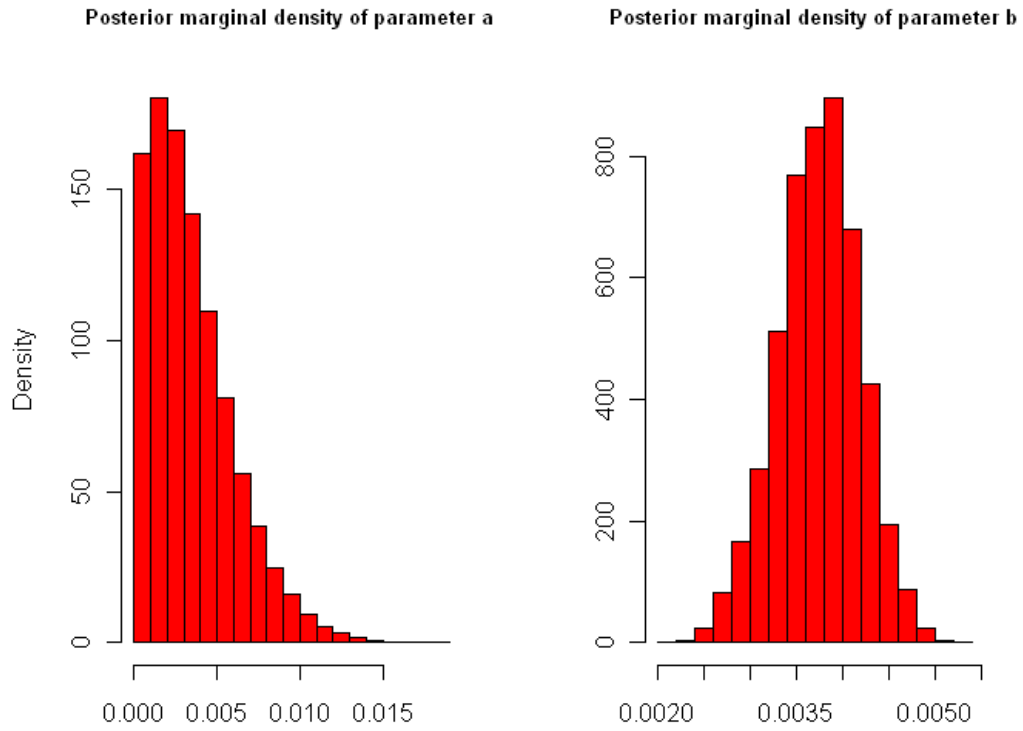

Figure 13.1.5 Posterior marginal density estimates of linear model parameters

Age-dependent evolution of uncertainties, present in failure rate estimates

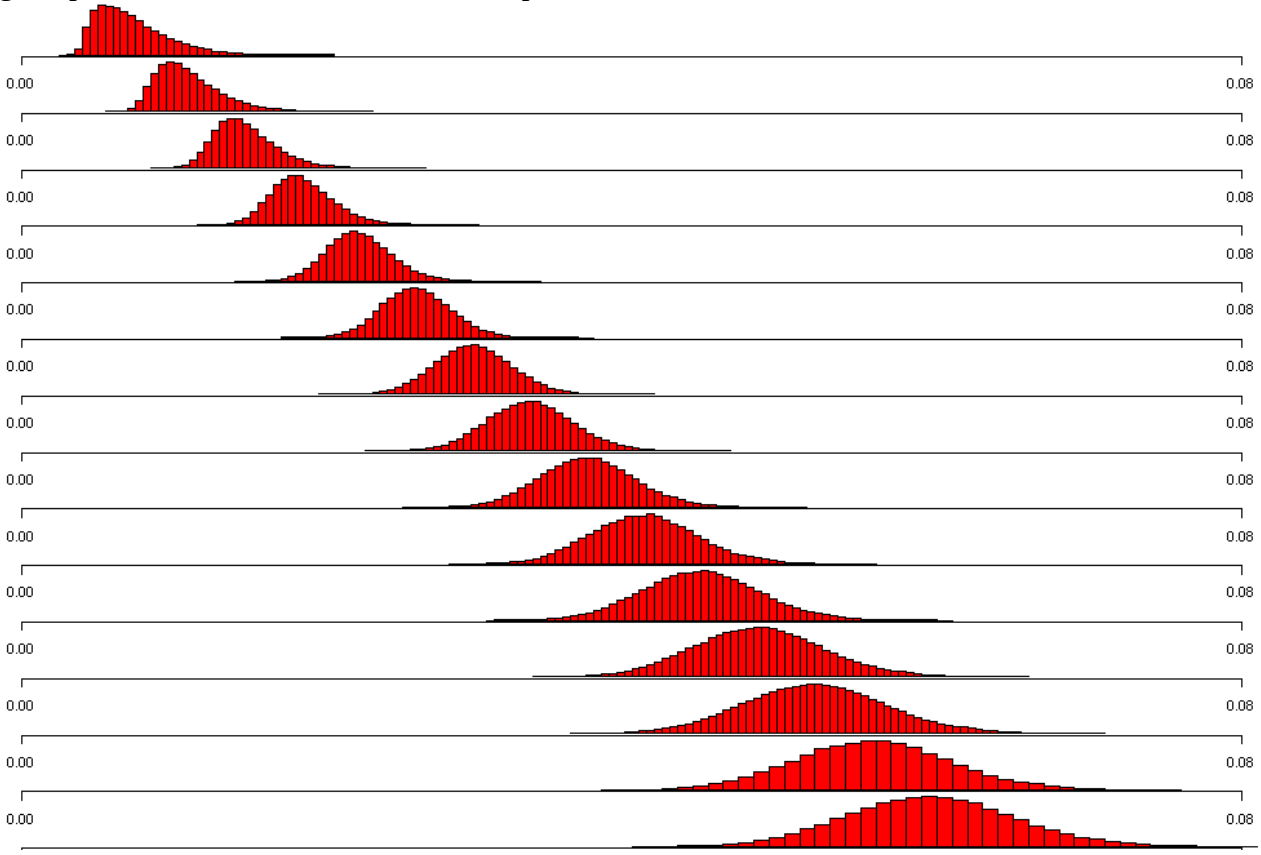

Figure 13.1.6 Age-dependent evolution of uncertainties of linear trend failure rate estimates

3. Exponential model: $\lambda(t) = e^{a+bt}$;

Posterior joint distribution of parameters a and b:

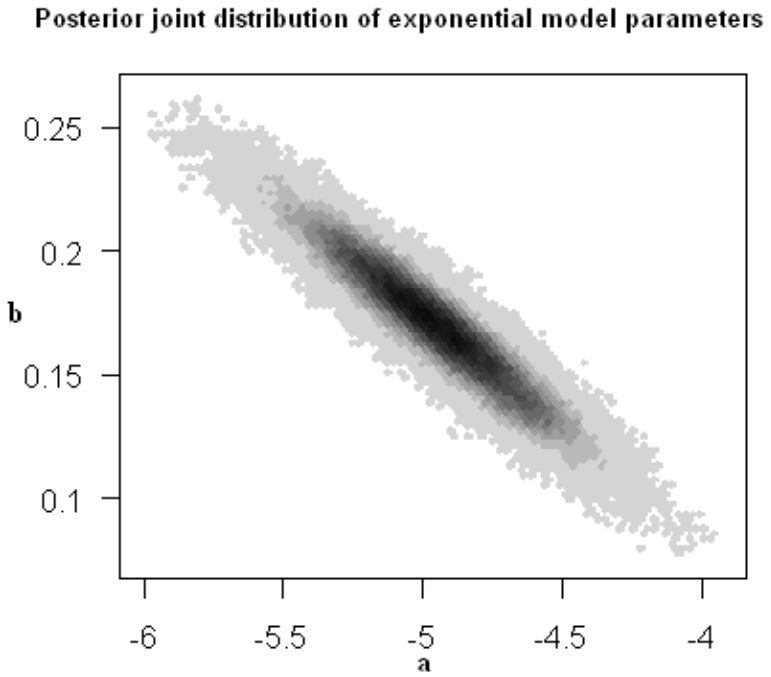

Figure 13.1.7 Joint distribution of exponential model parameters

Posterior marginal distribution of model parameters:

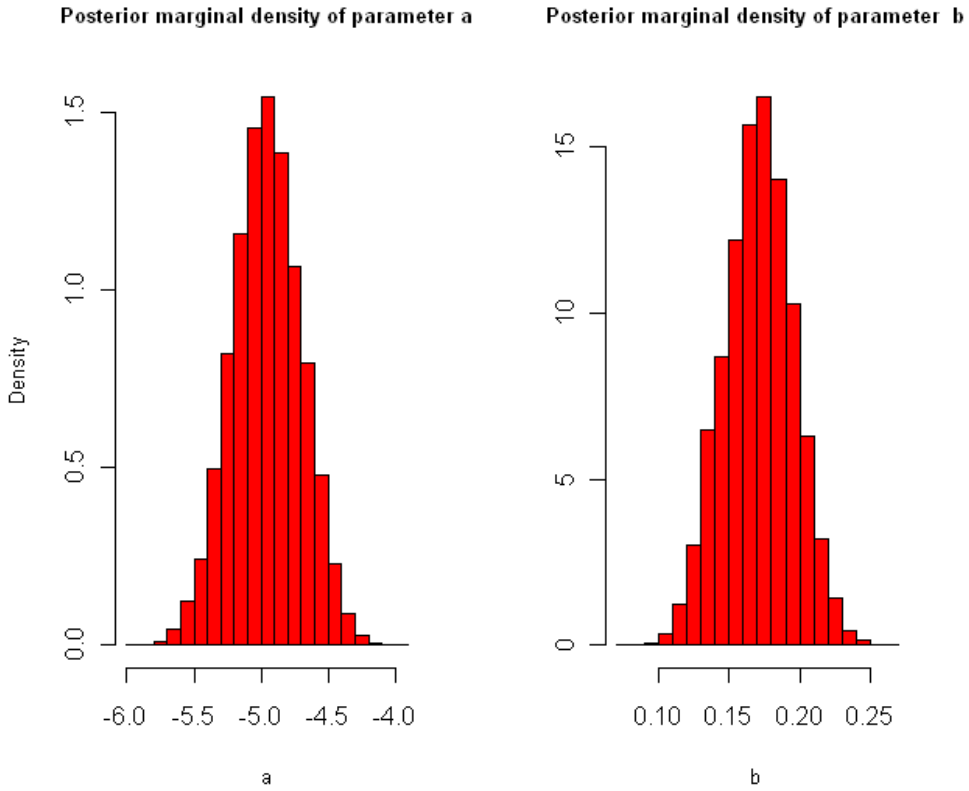

Figure 13.1.8 Posterior marginal density estimates of exponential model parameters

Age-dependent evolution of uncertainties, present in failure rate estimates

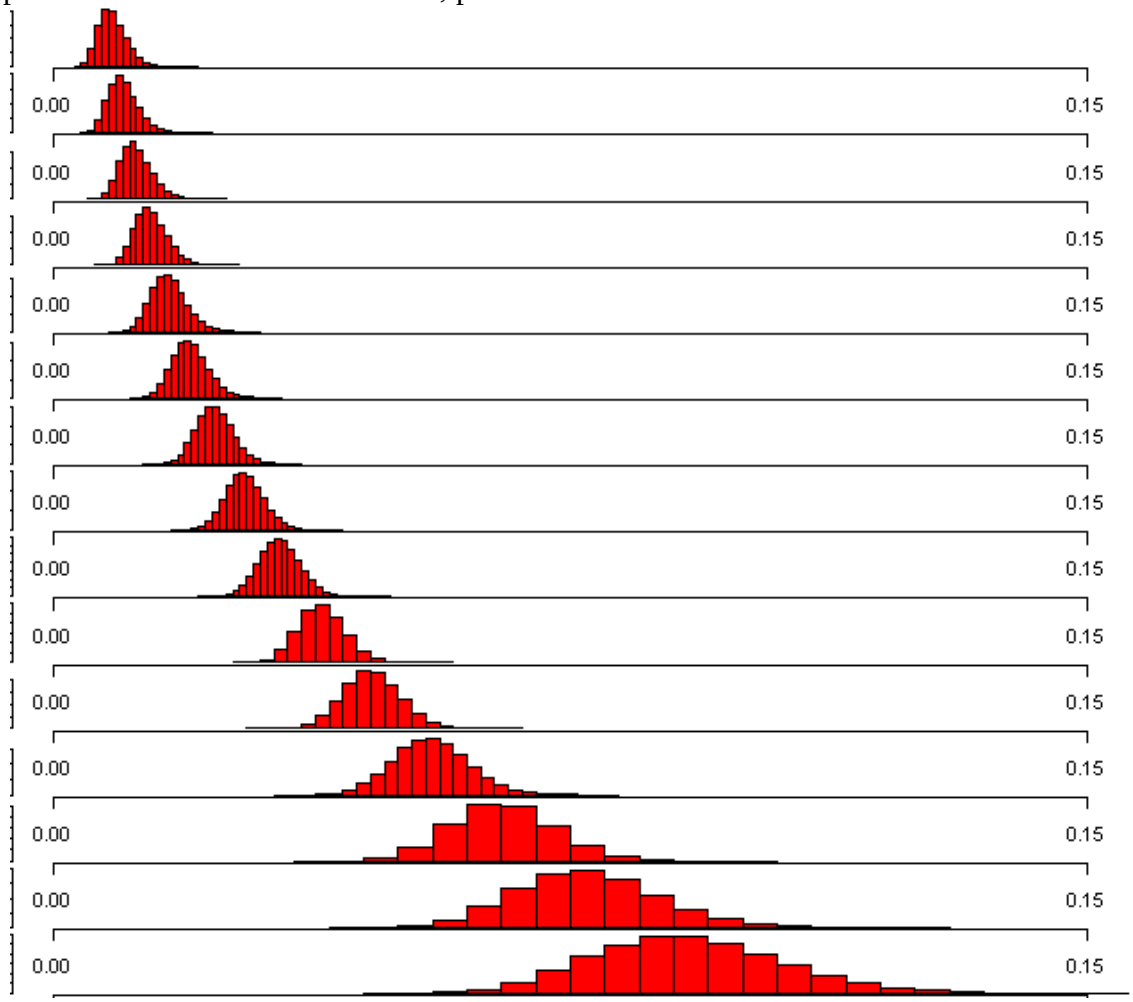

Figure 13.1.9 Age-dependent evolution of uncertainties of exponential trend failure rate estimates

4. Xie & Lai model: $\lambda(t) = at^{b-1} + ct^{d}$;

Posterior joint distribution of parameters a and b:

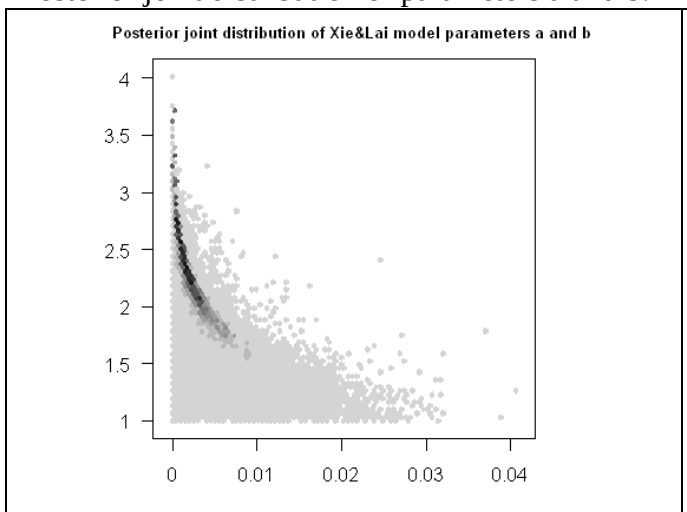

Figure 13.1.10 Joint distribution of Xie&Lai model parameters a and b

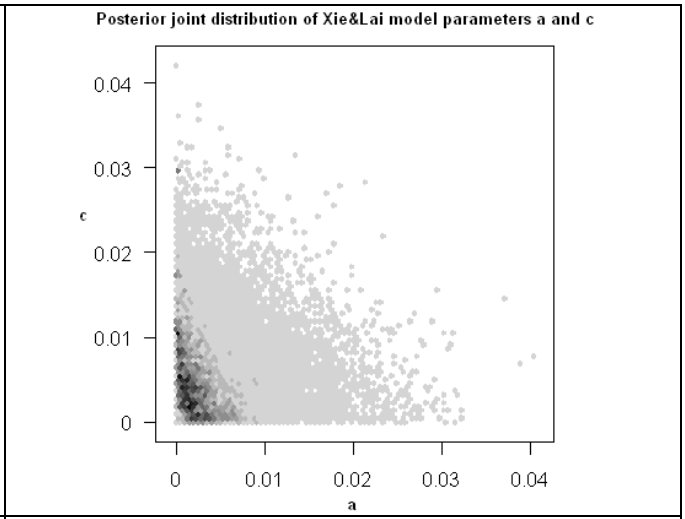

Figure 13.1.11 Joint distribution of Xie&Lai model parameters a and b

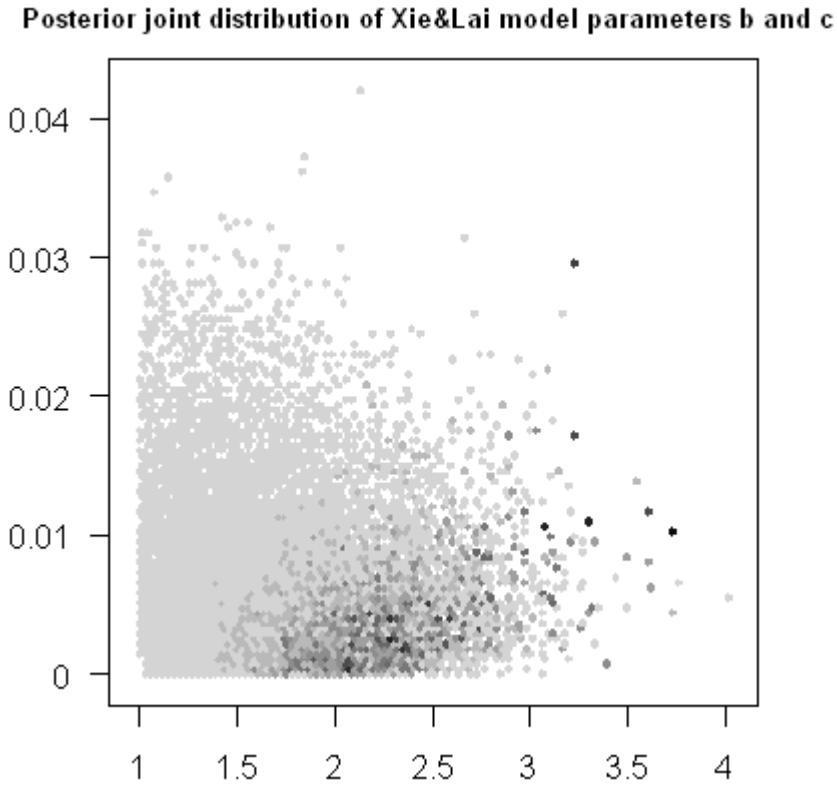

Figure 13.1.12 Joint distribution of Xie&Lai model parameters b and c

Posterior marginal distribution of model parameters:

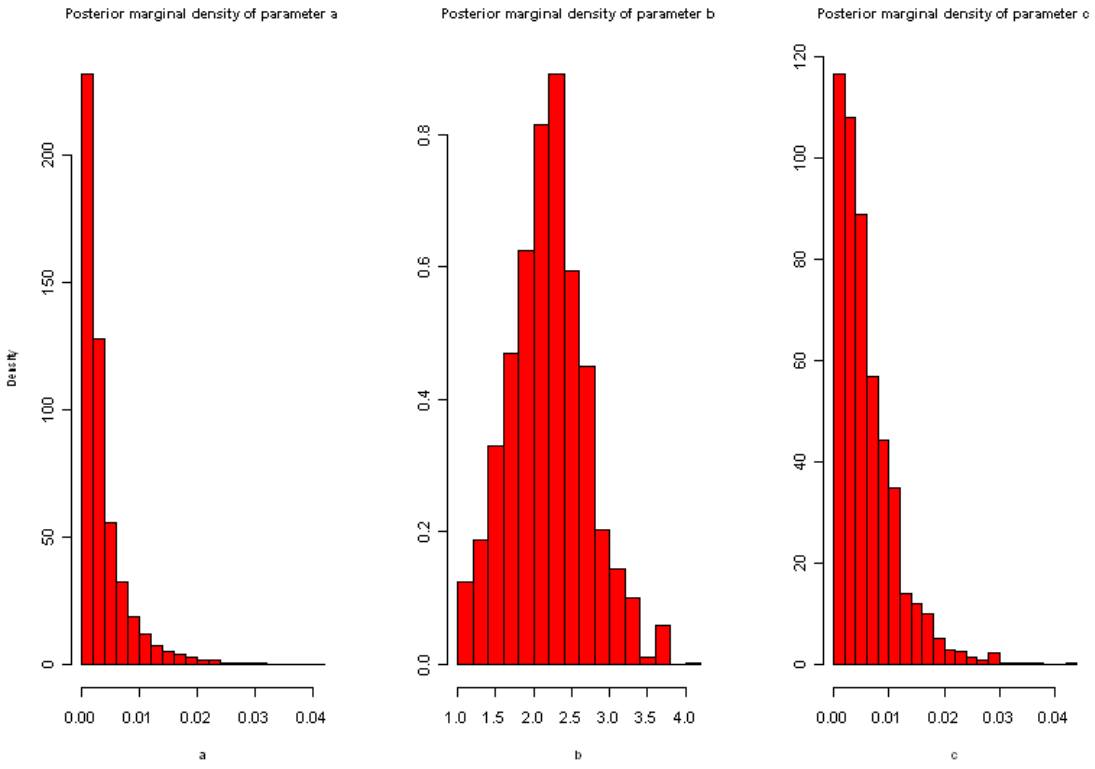

Figure 13.1.13 Posterior marginal density estimates of exponential model parameters

Age-dependent evolution of uncertainties, present in failure rate estimates

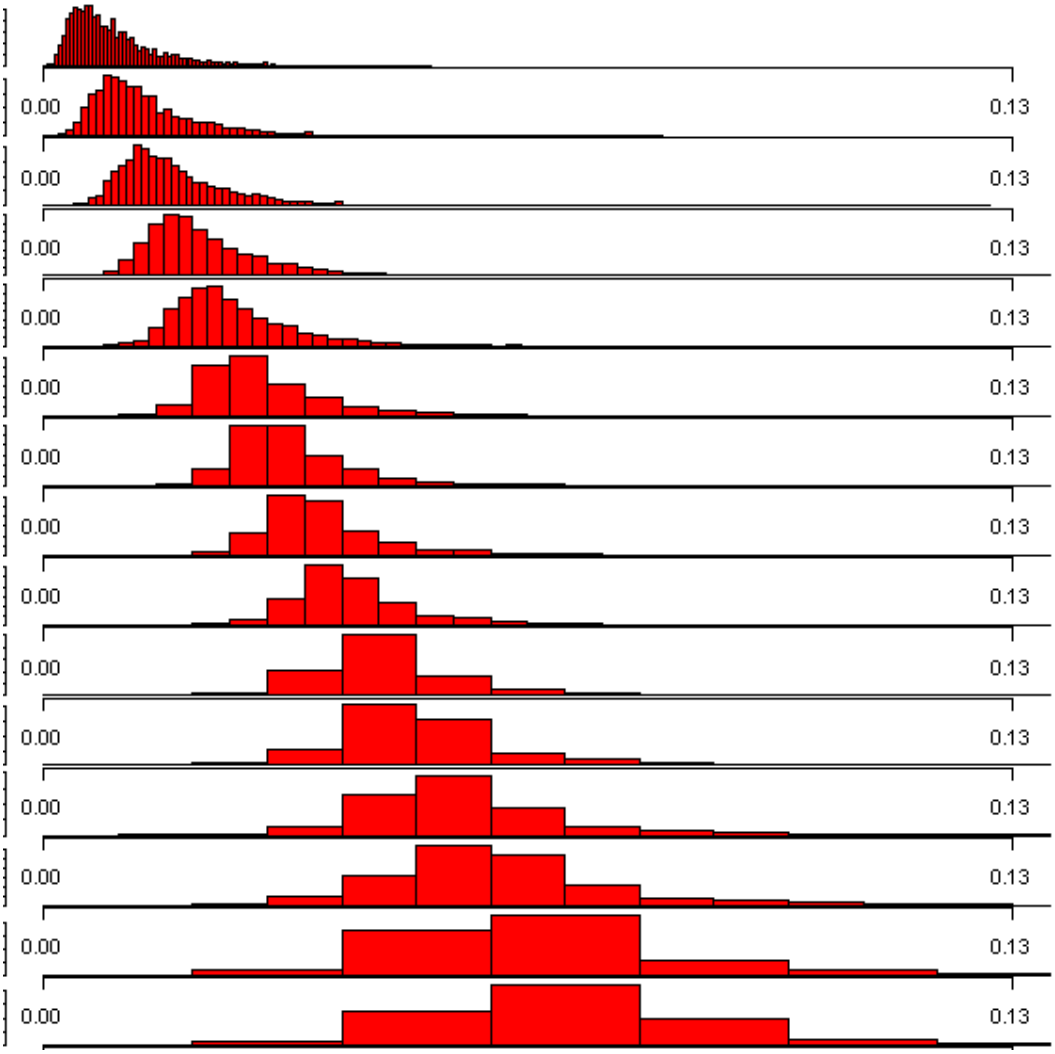

Figure 13.1.14 Age-dependent evolution of uncertainties of Xie&Lai trend failure rate estimates

13.2 CODES OF R AND WINBUGS PROGRAMS

Likelihood function of piecewise homogeneous Poisson process with Xie &Lai trend function (it is trivial to modify the code to adapt it to other cases of trend model).

```
loglike=function(a,b,c,d){
  t=c(1:15)
  lamb=a*(t^(b-1))+c*t^(d)
  sand=data[,2]*lamb
  suma=sum(data[,1]*log(sand))-sum(sand)-sum(lfactorial(data[,1]))
  suma
}
```

Code for 4 parameter Bayesian model with Adaptive Metropolis sampling:

```
n=4*300000; X=rep(0,n); X=matrix(data=X,nrow=4,ncol=(n/4),byrow=TRUE)
X[,1]=runif(4,min=0.01,max=0.1)
```

#variances of parameters

```
C=matrix(0.00001,4,4)
C[1,1]=0.01; C[2,2]=0.01; C[3,3]=0.01; C[4,4]=0.01

T0=1000
ST=rep(0,n)
ST=matrix(ST,nrow=1,ncol=(n/4),byrow=TRUE)
Rho=ST
```

#Algorithm for non-adaptive method part

```
for(i in 2:T0){
  Y=mvrnorm(1,X[,i-1],C)
  while((sum(Y[1:4]<=0)+sum(Y[2]<1)+sum(Y[4]>=1))>=1){Y=mvrnorm(1,X[,i-1],C)}
  rho1=loglike(Y[1],Y[2],Y[3],Y[4])+mvdnorm(X[,i-1],Y,C)
  rho1=rho1-loglike(X[1,i-1],X[2,i-1],X[3,i-1],X[4,i-1])
  rho1=rho1-mvdnorm(Y,X[,i-1],C)
  X[,i]=X[,i-1]+(Y-X[,i-1])*(log(runif(4))<rho1[1,1])
}
```

 # Set-up for adaptive method part

```
g1<-C
mean11<-rowMeans(X[,1:(T0-1)])
mean12<-rowMeans(X[,1:(T0)])

for(i in (T0+1):(100000)){
  Y=mvrnorm(1,X[,i-1],1*(g1+0.00001*diag(4)))
  while((sum(Y[1:4]<=0)+sum(Y[2]<1)+sum(Y[4]>=1))>=1){Y=mvrnorm(1,X[,i-
        1],1*(g1+0.00001*diag(4)))}
  rho1=loglike(Y[1],Y[2],Y[3],Y[4])+mvdnorm(X[,i-1],Y,1*(g1+0.00001*diag(4)))
```

```
   rho1=rho1-loglike(X[1,i-1],X[2,i-1],X[3,i-1],X[3,i-1])
   rho1=rho1-mvdnorm(Y,X[,i-1],1*(g1+0.00001*diag(4)))
   X[,i]=X[,i-1]+(Y-X[,i-1])*(log(runif(4))<rho1[1,1])
   mean12<-((i-1)/i)*mean12+(X[,i])/i
   mean11<-((i-2)/(i-1))*mean11+(X[,i-1])/(i-1)
   g1<-((i-2)/(i-1))*g1+((mean11)%*%t(mean11))+((X[,i]%*%t(X[,i])))/(i-1)-(i/(i-
         1))*((mean12)%*%t(mean12))
ST[i]=sum(g1)
}
```

WinBUGS code for Bayesian mode with Makeham trend function:

```
model{
for(i in 1:N){
     x[i]~dpois(mean1[i])
     x.rep[i]~dpois(mean1[i])
    mean1[i]<-lambda[i]*time[i]
    lambda[i]<-p[1]/(1+i)+p[2]*exp(p[3]*i)
    dummy[i]<-T[i]
}


############################
#chi-square discrepancy
############################

for(j in 1:N){
  chisq.rep[j]<-pow(x.rep[j]-mean1[j],2)/mean1[j]
  chisq.obs[j]<-pow(x[j]-mean1[j],2)/mean1[j]
}
chisq<-step(sum(chisq.rep[])-sum(chisq.obs[]))

############################
#standard deviation discrepancy
############################

stand<-step(sd(x.rep[])-sd(x[]))



for (i in 1:N){
  log.like[i]<--mean1[i]+x[i]*log(mean1[i])-logfact(x[i])
}
exp.like<-exp(-sum(log.like[]))

p[1]~dunif(0,100)
p[2]~dunif(0,100)
p[3]~dunif(0,100)
}

T[]      x[]     time[]
1        1      126.56
```

```
2        1      171.62
3        3      231.36
4        1      314.8
5        10     396.6
6        8       400
7        16     396.76
8        11      380
9        12      363.34
10       8       336.73
11       16      281.68
12       9      273.42
13       10       288.44
14       16     168.58
15       15     85.16

END

list(N=15)
list(p = c(0.02,0.0065,0.18),x.rep=c(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0))
```

WinBUGS code for posterior Bayesian model averaging procedure:

```
model{

for(i in 1:N){
   x_mak[i]<-x[i]
   x_exp[i]<-x[i]
   x_lin[i]<-x[i]
   x_pow[i]<-x[i]
   x_xl[i]<-x[i]
}

for(i in 1:N){
     x_xl[i]~dpois(mean_xl[i])
     #time[] tai isdirbis
    mean_xl[i]<-lambda_xl[i]*time[i]
    #pacios intensyvumo fukcijos apibrezimas
   lambda_xl[i]<-p_xl[1]*pow(i,p_xl[2]-1)+p_xl[3]*pow(i,p_xl[4])
   dummy[i]<-T[i]
}
p_xl[1]~dunif(0,100)
p_xl[2]~dunif(1,10)
p_xl[3]~dunif(0,100)
p_xl[4]~dunif(0,1)



for(i in 1:N){
     x_exp[i]~dpois(mean_exp[i])
    mean_exp[i]<-lambda_exp[i]*time[i]
   lambda_exp[i]<-exp(p_exp[1]+p_exp[2]*i)
```

```
}
p_exp[1]~dunif(-10,3)
p_exp[2]~dunif(-10,3)

for(i in 1:N){
      x_lin[i]~dpois(mean_lin[i])
     mean_lin[i]<-lambda_lin[i]*time[i]
    lambda_lin[i]<-(p_lin[1]+p_lin[2]*i)

}
p_lin[1]~dunif(0,100)
p_lin[2]~dunif(0,100)

for(i in 1:N){
      x_pow[i]~dpois(mean_pow[i])
     mean_pow[i]<-lambda_pow[i]*time[i]
    lambda_pow[i]<-(p_pow[1]*pow(i,p_pow[2]))

}
p_pow[1]~dunif(0,100)
p_pow[2]~dunif(0,10)

for(i in 1:N){
      x_mak[i]~dpois(mean_mak[i])
     mean_mak[i]<-lambda_mak[i]*time[i]
    lambda_mak[i]<-p_mak[1]/(1+i)+p_mak[2]*exp(p_mak[3]*i)

}
p_mak[1]~dunif(0,100)
p_mak[2]~dunif(0,100)
p_mak[3]~dunif(0,10)

for(i in 1:N){
  lambda[i]<-
0.046*lambda_mak[i]+0.456*lambda_exp[i]+0.027*lambda_lin[i]+0.444*lambda_pow[i]+0.026
*lambda_xl[i]
}

}
```

T[]	x[]	time[]
1	1	126.56
2	1	171.62
3	3	231.36
4	1	314.8
5	10	396.6
6	8	400
7	16	396.76
8	11	380
9	12	363.34
10	8	336.73
11	16	281.68

```
12       9     273.42
13      10     288.44
14      16     168.58
15      15     85.16

END

list(N=15)

list(p_mak=c(0.02,0.0065,0.18))
list(p_exp=c(-4.95,0.17))
list(p_lin=c(0.0033,0.0038))
list(p_pow=c(0.0078,1.18))
list(p_xl=c(0.0027,2.322,0.01,0.445))
```