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# THE FRACTAL STRUCTURE OF ANALYTICAL SOLUTIONS TO FRACTIONAL RICCATI EQUATION

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## Abstract

Analytical solutions to the fractional Riccati equation are considered in this paper. Solutions to fractional differential equations are expressed in the form of fractional power series in the Caputo algebra. It is demonstrated that solutions to higher-order Riccati fractional equations inherit some solutions from lower-order Riccati equations under special initial conditions. Such nested

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and fractal-like structure of solutions is investigated by means of analytical fractional differentiation operator techniques and computational experiments.

*Keywords*: Fractional Differential Equation; Operator Calculus; Analytical Solution.



## 1. INTRODUCTION

Though the concept of fractional-order derivatives dates back to the 17th century, fractional-order calculus has become more prominently used for the modeling of real-world phenomena only in recent years.[1,2] Extensive applications of such models can be encountered in the fields of physics,[3] engineering,[4] biomedicine[5] and image processing.[6] A short review of typical examples concerning the use of fractional calculus in mathematical modeling is given below.

A novel fractional differential and variational model capable of realizing the image fusion, super-resolution, and the edge information enhancement simultaneously has been introduced in Ref. 7. A new framework of nonlocal deformation in non-rigid image registration is developed using fractional Euler–Lagrange equations in Ref. 8. A spatial fractional telegraph equation is used to construct an algorithm for image structure preserving denoising in Ref. 9.

The use of matrix fractional differential equations in economic and quantum physics has been discussed in Ref. 10. It is shown in Ref. 11 that fractional-order models are better suited than their integer counterparts in modeling the properties of electrical energy storage devices. A moderate epidemiological model is used for the description of computer viruses with a fractional-order derivative having non-singular kernel in Ref. 12. It was demonstrated in Ref. 13 that fractional convection–diffusion equations can capture the gas breakthrough curves including their apparent positive skewness.

A spatial fractional-order thermal transport equation with the Caputo fractional derivative is proposed in Ref. 14 to describe convective heat transfer of nanofluids within disordered porous media in boundary layer flow. Viscoelastic constitutive laws for arterial wall mechanics are investigated using fractional-order partial differential equations (PDEs) in Ref. 15. A novel variable order fractional differential-based texture enhancement algorithm with applications used in medical imaging is developed in Ref. 16. High-order fractional PDEs are applied to the surface generation of proteins in Ref. 17.

Operator-based approach for the construction of analytic solutions to fractional differential equations is reported in Ref. 18. This technique is based on Caputo algebra of fractional power series and fractional differentiation and integration operators defined on the basis of this algebra. The main objective of this paper is to investigate the fractal structure of such analytic solutions to FDEs.

## 2. PRELIMINARIES

Main concepts and definitions concerning Caputo fractional power series and operators are presented in this section. The fractional power series presented here are a generalization of Ref. 18. Note that Caputo fractional differentiation and integration are defined differently than in the classical sense (via integral transformations), but through series basis functions. However, these two approaches yield equivalent results.

### 2.1. Caputo Fractional Power Series and Operators

Let $\frac{1}{n}, n \in \mathbb{N}$ denote the order of the considered fractional derivative. Consider the following sequence of functions:

$$w_j^{(n)} = \frac{x^{\frac{j}{n}}}{\Gamma(\frac{j}{n}+1)}, \quad j = 0, 1, \ldots. \tag{1}$$

If the derivative order reads $\frac{m}{n}, m < n$, where $m, n$ are coprime natural numbers, then the basis is still defined as in (1).

The following fractional power series are considered in this paper:

$$f = \sum_{j=0}^{+\infty} c_j w_j^{(n)}, \quad c_j \in \mathbb{C}. \tag{2}$$

Series defined by (2) are called Caputo fractional power series. The set of all such series is denoted as ${}^{C}\mathbb{F}_n$. Addition and multiplication of series in this set is performed using conventional operations. Note that $w_l^{(n)} w_k^{(n)} = \binom{\frac{k+l}{n}}{\frac{k}{n}} w_{k+j}^{(n)}$. Given two fractional power series $f = \sum_{j=0}^{+\infty} c_j w_j^{(n)}$ and $g = \sum_{j=0}^{+\infty} b_j w_j^{(n)}$, the product is defined in the Cauchy sense:

$$f \cdot g = \sum_{j=0}^{+\infty} \left( \sum_{r=0}^{j} \binom{\frac{j}{n}}{\frac{r}{n}} c_r b_{j-r} \right) w_j^{(n)}, \tag{3}$$

where

$$\binom{\lambda}{\mu} = \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\lambda-\mu+1)}; \tag{4}$$

denotes the generalized binomial coefficient for $\lambda, \mu \in \mathbb{R}, \lambda \geq \mu$.

As proven in Ref. 18, the set ${}^C\mathbb{F}_n$ with addition, multiplication and product by a scalar operations forms an algebra ${}^C\mathcal{F}_n$ over the field $\mathbb{C}$.

For any function $f \in {}^C\mathbb{F}_n$, Caputo integration and differentiation operators are defined by the following equalities:

$${}^C\mathbf{I}^{(1/n)} w_j^{(n)} = w_{j+1}^{(n)}, \quad j = 0, 1, \ldots; \tag{5}$$

$${}^C\mathbf{D}^{(1/n)} w_j^{(n)} = \begin{cases} 0, & j = 0; \\ w_{j-1}^{(n)}, & j = 1, 2, \ldots. \end{cases} \tag{6}$$

Fractional derivatives and integrals of order $\frac{m}{n}$ are represented by powers of the respective operators: $({}^C\mathbf{I}^{(1/n)})^m, ({}^C\mathbf{D}^{(1/n)})^m$.

## 3. SOLUTION OF THE FRACTIONAL RICCATI EQUATION

As mentioned in Sec. 1, the main objective of this paper is to explore the fractal structure of analytic solutions expressible in the form of Caputo fractional power series. Without loss of generality, the following fractional Riccati equation is considered:

$$({}^C\mathbf{D}^{(1/n)})^n y_n = a_2 y_n^2 + a_1 y_n + a_0, \tag{7}$$

where $a_k \in \mathbb{C}$ and $y_n = y_n(x; s_0^{(n)}, s_1^{(n)}, \ldots, s_{n-1}^{(n)}) \in {}^C\mathcal{F}_n$. The parameters $s_0^{(n)}, \ldots, s_{n-1}^{(n)}$ correspond to initial conditions formulated at $x = 0$:

$$({}^C\mathbf{D}^{(1/n)})^k y_n\big|_{x=0} = s_k^{(n)}; \quad k = 0, \ldots, n-1. \tag{8}$$

As noted in the previous section, if non-integers powers of the series $y_n$ are considered, the operator $({}^C\mathbf{D}^{(1/n)})^n$ is not identical to $\frac{\mathrm{d}}{\mathrm{d}x}$, because the former operator is applied to the Caputo power series comprised of basis elements $w_0^{(n)}, w_1^{(n)}, \ldots,$ while the latter acts on power series containing only integer powers of $x$.

In the remainder of this section, the solution to (7) is derived by computing first the series coefficients of the solution $y_n$. Once the recursive relation that defines the series coefficients is known, a generating function for these coefficients can be defined via an ordinary differential equation. The solution to this equation is a transformation of the solution to (7).

### 3.1. Derivation of Recurrence Relations for the Coefficients of the Solution to the Fractional Riccati Equation

Since $y_n \in {}^C\mathcal{F}_n$, it has the power series form:

$$y_n = \sum_{j=0}^{+\infty} c_j w_j^{(n)}; \quad c_j \in \mathbb{C}. \tag{9}$$

The definition of operators ${}^C\mathbf{D}^{(1/n)}$ and conventional algebraic operations with power series yields the following identities:

$$({}^C\mathbf{D}^{(1/n)})^n y_n = \sum_{j=0}^{+\infty} c_{j+n} w_j^{(n)}; \tag{10}$$

$$y_n^2 = \sum_{j=0}^{+\infty} \left( \sum_{r=0}^{j} \binom{\frac{j}{n}}{\frac{r}{n}} c_r c_{j-r} \right) w_j^{(n)} = \sum_{j=0}^{+\infty} \left( \sum_{r=0}^{j} \frac{\Gamma(\frac{j}{n}+1)}{\Gamma(\frac{r}{n}+1)\Gamma(\frac{j-r}{n}+1)} c_r c_{j-r} \right) w_j^{(n)}. \tag{11}$$

Inserting (9)–(11) into (7) and collecting like terms will result in

$$\sum_{j=0}^{+\infty} c_{j+n} w_j^{(n)} = \sum_{j=0}^{+\infty} \left( a_2 \left( \sum_{r=0}^{j} \frac{c_r}{\Gamma\left(\frac{r}{n}+1\right)} \frac{c_{j-r}}{\Gamma(\frac{j-r}{n}+1)} \right) \cdot \Gamma\left(\frac{j}{n}+1\right) + a_1 c_j + \delta_j a_0 \right) w_j^{(n)}, \tag{12}$$

where $\delta_j = 1$ if $j = 0$ and zero otherwise.

Equation (12) yields a recursive relation for the coefficients of the solution $y_n$:

$$c_{j+n} = a_2 \left( \sum_{r=0}^{j} \frac{c_r}{\Gamma(\frac{r}{n}+1)} \frac{c_{j-r}}{\Gamma(\frac{j-r}{n}+1)} \right) \times \Gamma\left(\frac{j}{n}+1\right) + a_1 c_j + \delta_j a_0, \quad (13)$$

for $j = 0, 1, \ldots$. To simplify (13), the following transformation is used:

$$\gamma_j^{(n)} = \frac{c_j}{\Gamma(\frac{j}{n}+1)}; \quad j = 0, 1, \ldots. \quad (14)$$

Rearranging (13) results in

$$(j+n)\gamma_{j+n}^{(n)} = n\left( a_2 \sum_{r=0}^{j} (\gamma_r^{(n)} \gamma_{j-r}^{(n)}) + a_1 \gamma_j^{(n)} + \delta_j a_0 \right); \quad j = 0, 1, \ldots. \quad (15)$$

Note that the first $n$ coefficients $\gamma_0^{(n)}, \ldots, \gamma_{n-1}^{(n)}$ can be chosen arbitrarily. Thus, the series $y_n$ is made to conform to initial conditions (8). Since the definition of operator ${}^{C}\mathbf{D}^{(1/n)}$ yields that $c_k = s_k^{(n)}; k = 0, 1, \ldots, n-1$, the following relation holds:

$$s_k^{(n)} = \Gamma\left(\frac{k}{n}+1\right)\gamma_k^{(n)}, \quad k = 0, 1, \ldots, n-1. \quad (16)$$

Thus, for a given set of initial conditions, the first $n$ coefficients of recurrence sequence $\gamma_j^{(n)}$ is computed using $s_0^{(n)}, \ldots, s_{n-1}^{(n)}$ and further iterated via the formula (15).

### 3.2. Characteristic Function of the Sequence $(\gamma_j^{(n)}; j = 0, 1, \ldots)$ and its Generating Equation

The characteristic function of sequence $(\gamma_j^{(n)}; j = 0, 1, \ldots)$ reads

$$\varphi_n(t) = \sum_{j=0}^{+\infty} \gamma_j^{(n)} t^j. \quad (17)$$

Multiplying both sides of (15) by $t^j$ and summing from $j = 0$ results in the equality:

$$\sum_{j=0}^{+\infty} (j+n)\gamma_{j+n}^{(n)} t^j = n\left( a_2 \sum_{j=0}^{+\infty} \left( \sum_{r=0}^{j} \left( \gamma_r^{(n)} \gamma_{j-r}^{(n)} \right) t^j \right) + a_1 \sum_{j=0}^{+\infty} \gamma_j^{(n)} t^j + a_0 \right). \quad (18)$$

Applying (17) to (18) yields

$$\sum_{j=n}^{+\infty} j\gamma_j^{(n)} t^{j-n} = n(a_2\varphi_n^2(t) + a_1\varphi_n(t) + a_0). \quad (19)$$

Note that the left-hand side of (19) can be rewritten as

$$\sum_{j=n}^{+\infty} j\gamma_j^{(n)} t^{j-n} = \frac{1}{t^{n-1}} \left( \sum_{j=1}^{+\infty} j\gamma_j^{(n)} t^{j-1} - \sum_{j=1}^{n-1} j\gamma_j^{(n)} t^{j-1} \right) = \frac{1}{t^{n-1}} \left( \frac{\mathrm{d}\varphi_n}{\mathrm{d}t} - \sum_{j=1}^{n-1} j\gamma_j^{(n)} t^{j-1} \right). \quad (20)$$

Inserting (20) into (19) and simplifying yield an ordinary differential equation with respect to the generating function $\varphi_n$:

$$\frac{\mathrm{d}\varphi_n}{\mathrm{d}t} = nt^{n-1}(a_2\varphi_n^2(t) + a_1\varphi_n(t) + a_0) + \sum_{j=1}^{n-1} j\gamma_j^{(n)} t^{j-1}, \quad n = 2, 3, \ldots. \quad (21)$$

Note that for the non-fractional Riccati equation ($n = 1$), Eq. (21) and the Riccati equation itself coincide.

### 3.3. Solution of the Fractional Riccati Equation via Generating Function $\varphi_n$

The function $\varphi_n(t)$ can be utilized to express solutions to (7). First, note that $c_j = \Gamma(\frac{j}{n}+1)\gamma_j^{(n)}, j = 0, 1, \ldots$. Then $y_n$ can be written as

$$y_n = \sum_{j=0}^{+\infty} c_j w_j^{(n)} = \sum_{j=0}^{+\infty} \Gamma\left(\frac{j}{n}+1\right)\gamma_j^{(n)} \frac{x^j n}{\Gamma(\frac{j}{n}+1)} = \sum_{j=0}^{+\infty} \gamma_j^{(n)} x^{\frac{j}{n}} = \varphi_n(\sqrt[n]{x}). \quad (22)$$

Also, note that

$$\frac{\mathrm{d}y_n}{\mathrm{d}x} = \frac{\mathrm{d}\varphi_n}{\mathrm{d}t}\bigg|_{t=\sqrt[n]{x}} \cdot \frac{1}{n}x^{\frac{1-n}{n}}, \tag{23}$$

which leads to

$$\frac{\mathrm{d}\varphi_n}{\mathrm{d}t}\bigg|_{t=\sqrt[n]{x}} = nx^{\frac{n-1}{n}}\frac{\mathrm{d}y_n}{\mathrm{d}x}. \tag{24}$$

Evaluating both sides of (21) at $t = \sqrt[n]{x}$ yields

$$nx^{\frac{n-1}{n}}\frac{\mathrm{d}y_n}{\mathrm{d}x} = nx^{\frac{n-1}{n}}(a_2y_n^2 + a_1y_n + a_0) + \sum_{j=1}^{n-1} j\gamma_j^{(n)}x^{\frac{j-1}{n}}, \tag{25}$$

that can be further simplified into

$$nx^{\frac{n-1}{n}}\left(\frac{\mathrm{d}y_n}{\mathrm{d}x} - a_2y_n^2 - a_1y_n - a_0\right) = \sum_{j=1}^{n-1} j\gamma_j^{(n)}x^{\frac{j-1}{n}}. \tag{26}$$

Equation (26) yields the following result.

**Remark.** Let $\psi_n(x) := \sum_{j=0}^{n-1}\gamma_j^{(n)}x^{\frac{j}{n}}$. Then, the initial value problem on the Riccati fractional differential equation (7), (8) is equivalent to the initial value problem on the following ordinary differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}(y_n - \psi_n) = a_2y_n^2 + a_1y_n + a_0; \tag{27}$$

$$y_n(0) = s_0^{(n)} = \gamma_0^{(n)}. \tag{28}$$

**Corollary.** *Since the expressions for $y_n$ and $\psi_n$ are known,* (27) *and* (28) *can be rewritten as*

$$\sum_{j=n}^{+\infty}\frac{j}{n}\gamma_j^{(n)}x^{\frac{j-n}{n}} = \sum_{j=0}^{+\infty}\left(a_2\sum_{r=0}^{j}\gamma_r^{(n)}\gamma_{j-r}^{(n)} + a_1\gamma_j^{(n)}\right) + a_0. \tag{29}$$

*Note that* (29) *is equivalent to* (15).

It can be observed from (26) that some solutions to the fractional Riccati equation remain viable for any value of $n$. If the initial conditions are set to $s_1^{(n)} = s_2^{(n)} = \cdots = s_{n-1}^{(n)} = 0$, then the right-hand side of (26) vanishes and, furthermore $\gamma_j^{(n)} \neq 0$ only if $j = kn$ for some $k \in \mathbb{N}$. In that case, the solution $y_n \in {}^C\mathcal{F}_1$ and satisfies the ordinary Riccati equation:

$$\frac{\mathrm{d}y_n}{\mathrm{d}x} = a_2y_n^2 + a_1y_n + a_0. \tag{30}$$

This observation leads to the conclusion that for any $n \in \mathbb{N}$, the fractional Riccati equation (7) inherits the non-fractional solutions of (30) for some initial conditions. Furthermore, to showcase the fractal nature of fractional differential equations, this argument can be extended: any Eq. (7) with order $n = km, k, m \in \mathbb{N}$ inherits solution from the fractional equation with orders $n = k$ and $n = m$.

## 4. COMPUTATIONAL EXPERIMENTS

In this section, the fractal nature of the analyzed fractional differential equations will be demonstrated using numerical experiments. Let us consider the following fractional Riccati equation:

$$({}^C\mathbf{D}^{(1/n)})^n y_n = y_n^2 + y_n - 6. \tag{31}$$

A comparison of numerical integration results for the fractional Riccati differential equation (31) of different orders ($n = 1, 2, 3$) is presented below.

The non-fractional Riccati equation (31) with $n = 1$ admits only well-known kink solutions,[19,20] depicted in Fig. 1. Note that the number of initial conditions on (31) increases as $n$ grows, thus yielding a larger set of solutions. A comparison of solutions to the non-fractional and fractional Riccati equations (for $n = 1$ and $n = 2$, respectively) is shown in Fig. 2.

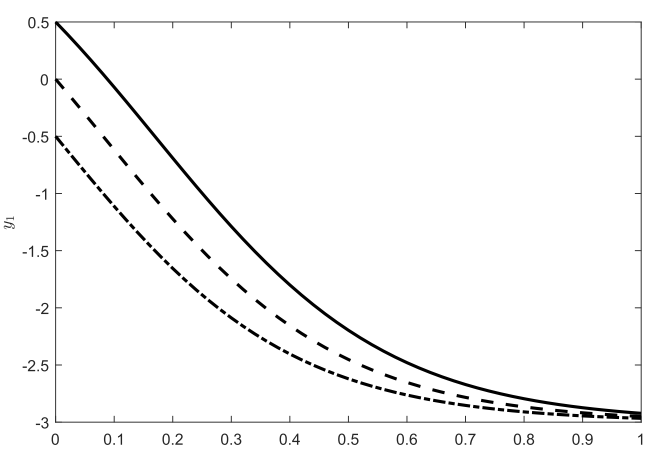


**Fig. 1** Kink solutions to the non-fractional Riccati equation (31) for $n = 1$. The solid, dashed and dash-dotted lines correspond to initial conditions $s_0^{(1)} = \frac{1}{2}, 0$ and $-\frac{1}{2}$, respectively.

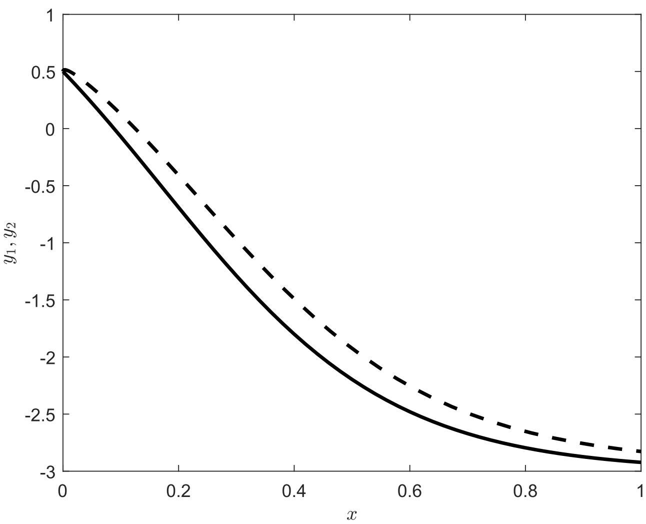


**Fig. 2** Solutions to Eq. (31) for $n = 1$ (solid line) and $n = 2$ (dashed line). Initial conditions are set to $s_0^{(1)} = s_0^{(2)} = s_1^{(2)} = \frac{1}{2}$.

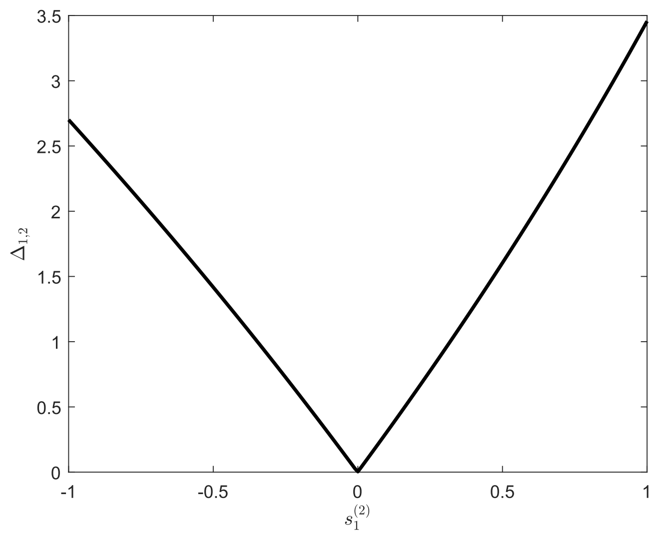


**Fig. 3** Plot of $\Delta_{1,2}(s_1^{(2)})$ for $s_0^{(1)} = s_0^{(2)} = \frac{1}{2}$. Note that the fractional and non-fractional solutions coincide when $s_1^{(2)} = 0$.

To compare numerical solutions to (31) with different values of $n$, the following difference measure is introduced:

$$\Delta_{n,m}\big(s_1^{(n)}, \ldots, s_{n-1}^{(n)}; s_1^{(m)}, \ldots, s_{m-1}^{(m)}\big) = \sum_{j=0}^{N} \big(\widehat{y}_n\big(jh; s_1^{(n)}, \ldots, s_{n-1}^{(n)}\big) - \widehat{y}_m\big(jh; s_1^{(m)}, \ldots, s_{m-1}^{(m)}\big)\big)^2, \quad (32)$$

where $\widehat{y}_k(x; s_1^{(k)}, \ldots, s_{k-1}^{(k)})$ is the numerical solution to (31) of order $k$ with initial conditions $s_1^{(k)}, \ldots, s_{k-1}^{(k)}$; $h$ is the constant integrator step-size (the classical Runge–Kutta fourth-order method was used in the computations); $N$ is the number of time-forward steps. Initial conditions $s_0^{(n)}$ and $s_0^{(m)}$ are set to be equal.

The plot of $\Delta_{1,2}(s_1^{(2)})$ when $s_0^{(1)} = s_0^{(2)} = \frac{1}{2}$ is shown in Fig. 3. It can be seen that $y_2$ coincides with the kink solution of the non-fractional Riccati equation when $s_1^{(2)} = 0$, which verifies the analytical results presented in the previous section.

An analogous experiment was performed to compare solutions of the non-fractional Riccati equation and fractional equation of order $n = 3$. The initial conditions were set to $s_0^{(1)} = s_0^{(3)} = \frac{1}{2}$. The contour plot for $\Delta_{1,3}(s_1^{(3)}, s_2^{(3)})$ is given in Fig. 4. It can be observed that the kink solution of the non-fractional equation satisfies the fractional equation of order $n = 3$, but only if initial conditions

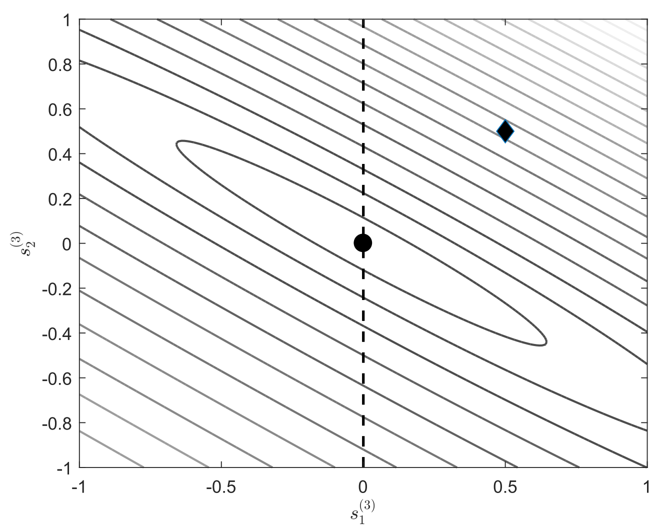

**Fig. 4** Contour plot of $\Delta_{1,3}(s_1^{(3)}, s_2^{(3)})$ for $s_0^{(1)} = s_0^{(3)} = \frac{1}{2}$. Solid lines indicate contour lines of $\Delta_{1,3}$; the black circle denotes the minimum point $s_1^{(3)} = s_2^{(3)} = 0$ where $\Delta_{1,3} = 0$; the dashed line corresponds to a plot of $\Delta_{1,3}(0, s_2^{(3)})$ shown in Fig. 5; the diamond corresponds to the initial conditions used to plot comparison of solutions $y_1$ and $y_3$ given in Fig. 6.

$s_1^{(3)} = s_2^{(3)}$ are equal to zero. This result is further clarified in Fig. 5, where the section of the contour plot along the line $s_1^{(3)} = 0$ is given. However, it must be noted that solutions to the non-fractional and fractional equations only coincide for a single pair of initial conditions $s_1^{(3)} = s_2^{(3)} = 0$. A plot of solutions $y_1, y_3$ for initial conditions not on the minimum point $(0, 0)$ is given in Fig. 6.

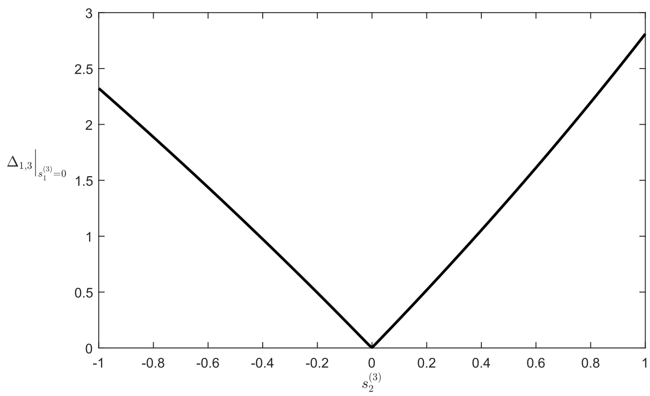


**Fig. 5** Plot of $\Delta_{1,3}(0, s_2^{(3)})$. Note that the solutions coincide on $s_2^{(3)} = 0$.

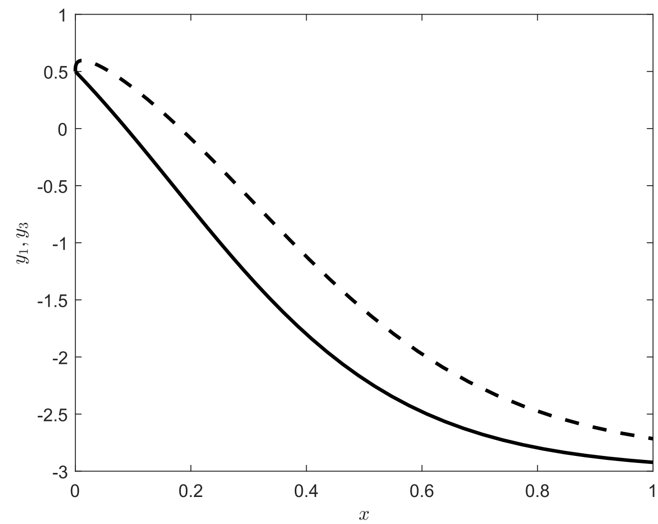


**Fig. 6** Plot of solutions $y_1$ (solid line) and $y_3$ (dashed line) for initial conditions $s_0^{(1)} = s_0^{(3)} = \frac{1}{2}$, $s_1^{(3)} = s_2^{(3)} = \frac{1}{2}$.

From the presented results, it follows that solutions of the fractional Riccati equations of orders $n = 2$ and $n = 3$ coincide for initial conditions $s_1^{(2)} = s_1^{(3)} = s_2^{(3)} = 0$. These results also hold true for higher order of fractional differential equations, which means that the non-fractional kink solution satisfies any fractional equation. Furthermore, a similar argument can be used to show that a fractional Riccati differential equation 7 of order $n$ admits all solutions (when some initial conditions are set to zero) of the same equation with order $m$ if $m$ divides $n$.

## 5. FRACTAL STRUCTURE OF ANALYTICAL SOLUTIONS TO FRACTIONAL RICCATI EQUATION

Computational experiments presented in the previous section indicate that the solutions to the fractional Riccati equation exhibit a nested, fractal-like structure in which solutions of lower-order

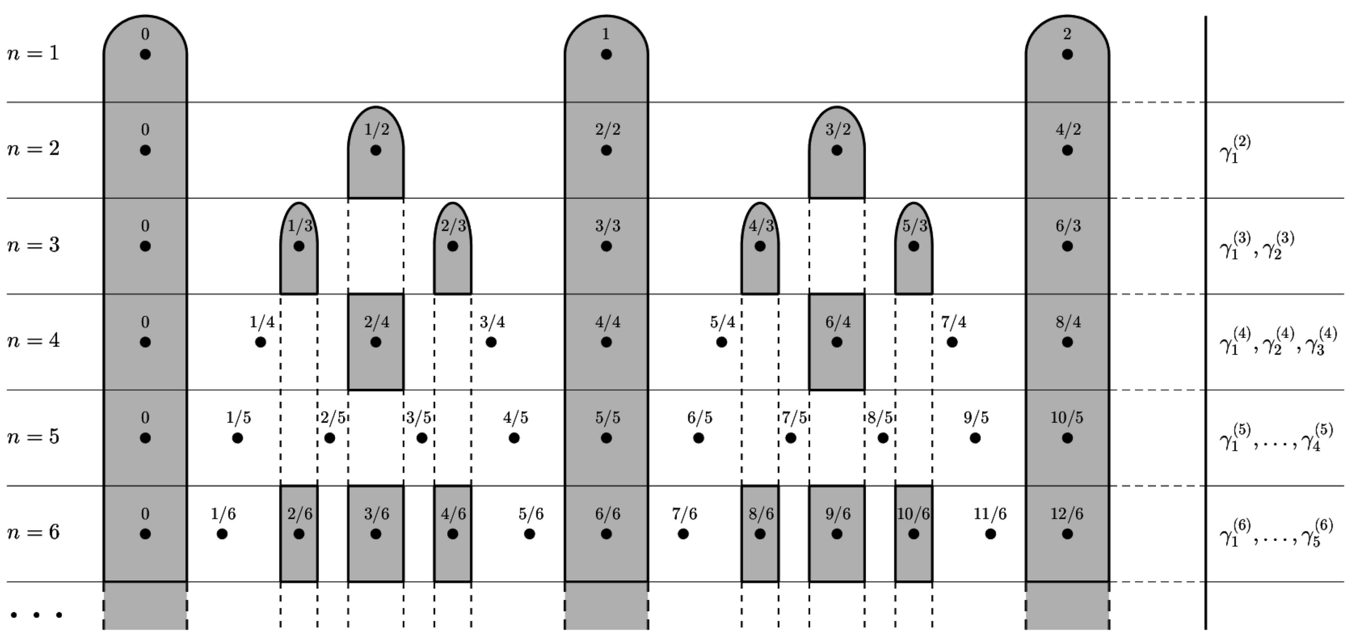


**Fig. 7** Fractal-like structure of fractional power series basis. Each row $n = k; k = 1, 2, \ldots$ displays basis elements $w_j^{(k)}; j = 0, 1, \ldots$ of the fractional power series of order $k$. Accordingly, fractions $\frac{j}{k}$ correspond to powers of $x$ for each base element $w_j^{(k)}; j = 0, 1, \ldots$, respectively. Parameters $\gamma_v^{(k)}; v = 1, \ldots, k-1$ on the right represent the set of arbitrarily chosen coefficients of the corresponding ODE (21). Gray-filled columns (or sections of columns) correspond to the same power of $x$ in respective base elements.

equations satisfy higher-order equations if some initial conditions are set to zero.

Let $p, q$ be relatively prime natural numbers. Consider Caputo power series algebras ${}^C\mathcal{F}_{pm}, {}^C\mathcal{F}_{qm}$, where $m \in \mathbb{N}$. The definition of fractional power series yields the following properties:

$$ {}^C\mathcal{F}_{pm} \cap {}^C\mathcal{F}_{qm} = {}^C\mathcal{F}_{m} \tag{33} $$

and

$$ {}^C\mathcal{F}_{pm} \cup {}^C\mathcal{F}_{qm} \subseteq {}^C\mathcal{F}_{pqm}. \tag{34} $$

The relationship between different orders of fractional power series is illustrated with more details in Fig. 7. It is clear that the basis elements corresponding to different orders $n$ of fractional differential equations may intersect. Thus, as demonstrated via numerical experiments in the previous section, solutions from a higher-order equation $q$ may inherit solutions from a lower-order equation $p < q$ under some initial conditions.

Consider two fractional Riccati equations (7) and (8) of orders $p$ and $q$. Let $g := \gcd(p, q)$ and define $s^{(p)} := \frac{p}{g}, s^{(q)} := \frac{q}{g}$. Note that $s^{(p)}, s^{(q)} \in \mathbb{N}$ and the basis of order $p, q$ intersect at powers of $x^{\frac{k}{g}}, k = 0, 1, \ldots$ if the following conditions on coefficients $\gamma_j^{(p)}, \gamma_j^{(q)}$ hold true:

$$ \gamma_j^{(p)} = 0, \quad j \neq s^{(p)} l; \tag{35} $$

$$ \gamma_i^{(q)} = 0, \quad i \neq s^{(q)} l, \tag{36} $$

where $l = 0, 1, \ldots$.

## 6. CONCLUSIONS

The fractal structure of solutions to fractional differential equations has been investigated in this paper. It is shown that the fractional Riccati equation can be solved by considering the recurrence relations between the coefficients of fractional power series. It is proven that the generating function of the series coefficients satisfies an associated ordinary differential equation and can be used to construct the solution to the fractional Riccati equation.

Furthermore, it appears that solutions to fractional equations exhibit a nested, fractal-like structure, which is investigated via computational experiments and theoretical investigation of the fractional power series basis. This fractality property results in the fact that higher-order fractional Riccati equations inherit some solutions from lower-order equations when a subset of initial conditions is set to zero.

Further investigation of the fractal properties and construction of analytical solutions of fractional differential equations remains a definite objective of future research.

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