

Article

Suitability of the Single Transferable Vote as a Replacement for Largest Remainder Proportional Representation

Martynas Patašius ^{1,2} 

¹ Department of Applied Informatics, Kaunas University of Technology, 44249 Kaunas, Lithuania; martynas.patasius@ktu.lt

² Institute of Biomedical Engineering, Kaunas University of Technology, 44249 Kaunas, Lithuania

Abstract: There are two main approaches to achieving proportional representation in elections: the single transferable vote and methods based on party lists. This paper discusses ways to use the single transferable vote while using some of the main features used with the largest remainder method, such as the electoral threshold. The investigation has shown that the Weighted Inclusive Gregory method is a suitable replacement for the largest remainder method when it is desirable to avoid wasted votes and to handle independent candidates in a straightforward way, but it is also desirable to keep the results as close to the ones achieved under the largest remainder method as possible. The investigation also led to the development of an algorithm for using the single transferable vote when preference lists are based on party lists, exploiting commonalities and symmetries between the patterns of preferences given in the votes. It has been shown that such an algorithm makes the calculations faster than the use of ordinary implementations of the single transferable vote when the numbers of seats and candidates are high, as commonly happens when methods based on party lists are used.

Keywords: single transferable vote; largest remainder proportional representation; electoral threshold

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1. Introduction

There are two main approaches used to achieve proportional representation in elections: the single transferable vote (STV) and methods based on party lists.

The most simple of proportional representation methods using party lists is the largest remainder method (LRM). Using this method, the votes given for each list are divided by the vote quota required for electing a single candidate. The quotients indicate the first part of the distributed mandates. The rest of the mandates are then distributed by their remainders in decreasing order.

Another group of proportional representation methods using party lists (including the d'Hondt method [1] and Sainte-Laguë method [2]) is based on division: the numbers of votes given for each list are divided by numbers chosen in a specific way, and the highest results are chosen, with a single candidate elected from each list for each division of results in the set of the highest results.

Approaches based on party lists also correspond to methods used for the apportionment of seats to electoral districts of smaller units. The largest remainder method, for example, corresponds to the Hamilton method, and the Sainte-Laguë method corresponds to the Weber method.

The single transferable vote is the family of systems where the votes are given as preference lists, and when a candidate is elected or eliminated, the surplus votes above the vote quota for elected candidates and all votes for eliminated candidates are transferred to the next candidate in the preference list [3]. A candidate is considered to be elected

after reaching the vote quota [3]. A candidate (usually the one with the least amount of first preferences, including transferred preferences) is eliminated if, at that iteration, no candidate has been elected [3]. If after some iteration a vote is eligible for transfer, but all candidates listed in it are elected or eliminated, the vote quota has to be recalculated [3].

Various methods of the single transferable vote use different means to transfer votes. In earlier versions, such votes were chosen at random, which was convenient for manual counting [3]. In later methods, surplus votes received by an elected candidate were transferred with a lower value as a proportion of the surplus to the total vote received by that candidate [3]. The first of such methods was the Gregory method, and further improvements include the Inclusive Gregory method and the Weighted Inclusive Gregory method (STV-WIG) [4]. Of those, the Gregory method included some simplifications to ease manual counting, which are removed in the Weighted Inclusive Gregory method; that is, in the Weighted Inclusive Gregory method, all votes received by the elected candidate are transferred to candidates listed after this elected candidate with a lower value as a proportion of the surplus to the total votes received by that elected candidate.

Meek's algorithm (STV-M) [5–8] assigns each candidate a weight (initially set to one) indicating the fraction of votes that the candidate keeps (The rest of the votes are transferred to further candidates). For the eliminated candidates, this is set to zero, and for the elected candidates, it is decreased until the weighted vote retained by the candidate is equal to the vote quota. Such recalculation is necessary whenever the candidate receives additional votes or the vote quota changes.

There are alternative ways to deal with the elimination of candidates, such as by using electability estimates [9], reintroducing previously eliminated candidates after a candidate has been elected [10], using trial runs to decide which candidates are to be eliminated [11–15] and by considering comparisons of pairs of candidates [3].

The single transferable vote is used in Ireland, Australia and Malta. Proportional representation using party lists is more common and is used, for example, in Austria, Belgium, Germany, Lithuania and Poland to name a few.

It is known that those methods should find similar results under some conditions [16], although it looks as though such conditions have not been investigated in more detail.

As has been seen, those methods tend to use vote quotas, and there are several ways to calculate vote quotas [17].

The Hare quota is expressed as follows:

$$q_h = \lfloor q_{hwr} \rfloor = \left\lfloor \frac{t}{w} \right\rfloor, \quad (1)$$

where t is the total number of votes, w is the number of candidates to be elected, $\lfloor \cdot \rfloor$ is the operation of rounding down and q_{hwr} is the Hare quota without rounding, which indicates the number of votes that each of the winning candidates would receive if all votes were equally distributed among them.

The Droop quota is expressed as follows:

$$q_d = \left\lfloor \frac{t}{w+1} + 1 \right\rfloor. \quad (2)$$

This indicates the minimal integer number of votes required to ensure election of the intended number of candidates.

The Droop quota without rounding would be

$$q_{dwr} = \frac{t}{w+1} + \varepsilon, \quad (3)$$

where ε is an arbitrary small but positive number.

It can be seen that the vote quota is a function, increasing (or at least not decreasing) as the total number of votes increases and decreasing (or at least not increasing) as the number of candidates to be elected increases.

The single transferable vote and proportional representation based on party lists are not merely mechanically different; they support various features.

For example, the single transferable vote easily supports the running of independent candidates, while proportional representation based on party lists requires various workarounds to support this conveniently. The most obvious method consists of treating individual independent candidates as party lists from a single candidate. However, that encourages strategic voting, as in such a case, it is inevitable that the votes above the amount necessary to elect the single candidate are going to be wasted (if a list were longer, surplus votes might lead to the election of other candidates), and encouragement of strategic voting is generally seen as undesirable.

Similarly, the single transferable vote easily supports having the same candidates supported by several parties, while proportional representation based on party lists requires the party lists to have no candidates in common.

The single transferable vote makes it easier to avoid wasted votes, as those votes not necessary to elect one candidate can be transferred to another candidate.

List-based proportional representation methods are used in multi-member districts of various sizes. The single transferable vote is used in districts electing a relatively small number of candidates. This is related to the fact that when using the single transferable vote, the voter is usually expected to rank the candidates one by one, which is hard to do if many candidates are to be elected. (Even in small districts, voters tend to avoid ranking all candidates, motivating investigations about the influence of truncations on the results [18]). It would also be hard to count the votes, as counting can sometimes take more than a week [18].

One of more important modifications easily supported by methods based on party lists is the electoral threshold (Th). When an electoral threshold is used, lists that receive less votes than the threshold are eliminated.

There are several ways in which one system can be made more similar to another. For example, the case of proportional representation based on party lists can be made slightly more like the single transferable vote using *apparentement*, whereby at one stage, a group of parties competes for seats together, and afterward, the parties share those seats among themselves [19].

Another proposed option is to let the voters rank not individual candidates but party lists [20].

H. R. Droop also already mentioned a pamphlet by Walter Baily which proposed letting candidates prepare the lists of preferences for vote transfer, with a later suggestion to let a group of voters prepare such a list as well [21].

Another option to use lists prepared beforehand is used in Australia, where a voter can choose to vote for a preference list proposed by a party instead of ranking the candidates. For example, in the Australian Senate election of 2004, 95.9% of voters voted for such party preference lists [22].

There was also a proposal to adapt the single transferable vote to the proportionality condition used for division methods [23].

One possible reason to look for methods of a single transferable vote that would be equivalent or similar to the methods of proportional representation based on party lists is that the legal requirements for an election method change sometimes. For example, in 2011, the Lithuanian Constitutional Court issued a decision stating that independent candidates have to have a possibility of participating in municipal elections (elections held using the largest remainder method) [24]. In another example, in 2014, the German Constitutional Court issued a decision declaring that a 3% electoral threshold for elections to the European Parliament was unconstitutional because it interfered with equal suffrage [25]. While it is known that it is impossible to create a voting method that has all features that are

considered to be desirable [26,27], it might still be worth looking for ways to meet those specific requirements.

When requirements for voting methods change, it can be assumed that the previously used method was seen to have some advantages (otherwise it would have been replaced by some other method already), and reluctance to lose those advantages can be expected. Therefore, it seems useful to explore ways to adapt the features of one method of proportional representation to another while trying to keep the results similar (therefore, hopefully, keeping the advantages in question).

Therefore, there is a reason to think that there might be a use for a voting method that, as in the largest remainder method, would be fast, able to deal with districts of various sizes and able to support electoral thresholds while, as in the single transferable vote method, being able to deal with independent candidates conveniently, making it possible to minimize wasted votes.

Thus, this paper discusses ways to use the single transferable vote method while keeping some of the main features used with the largest remainder method, such as the electoral threshold. This paper also discusses the conditions under which the single transferable vote method is equivalent to the largest remainder method, as while it is known that those methods are supposed to achieve similar results, it appears that the actual conditions for equivalence have not been investigated in detail.

By themselves, the findings of this paper are most easily applied to a voting method where voters choose among preference lists prepared beforehand and based on party lists. However, investigation of such special cases can be helpful for the investigation of more ordinary uses of the single transferable vote as well.

Based on such investigations, this paper also proposes an algorithm for using the single transferable vote method when preference lists are based on party lists, exploiting the commonalities and symmetries between the patterns of preferences given in the votes.

Naturally, investigation of the discussed methods, as with of other voting methods, can be expected to be useful for national and local elections. Still, it should not be forgotten that voting methods are also used in various organizations [28] and even in machine learning [29,30].

2. Equivalence between the Largest Remainder Method and STV

2.1. The Case of Preference Lists Coinciding with Party Lists

Here, we will use the largest remainder method, the Weighted Inclusive Gregory method (only with proportional decreasing of the number of votes instead of the calculation of the transfer value) and Meek's algorithm.

To simplify the further discussion, let us define the sufficiently long party list:

Definition 1. *Party lists are minimally sufficiently long when, if using the given methods, it would not be possible to increase the number of candidates elected from a given list by adding more candidates to it.*

Still, it can be seen that this definition is not very convenient, as it implies that an empty party list could be sufficiently long if it receives too few votes for the election of a single candidate:

Definition 2. *Party lists are sufficiently long if they are minimally sufficiently long and will stay minimally sufficiently long after removal of the last candidate.*

To make further analysis easier, let us prove several lemmas:

Lemma 1. *The results of the largest remainder method do not change if a candidate that has not been elected is removed.*

Proof. The largest remainder method results in the election of candidates that are the first on the lists which have assigned seats. Thus, it has to be shown that removal of a candidate that has not been elected cannot change the order of the other candidates in a way that would result in another candidate being elected and that it cannot result in a seat being assigned to another list.

As the first candidates are going to be elected, unelected candidates are going to be below them on the list. Thus, removal of any unelected candidate cannot result in the reordering of elected candidates.

The only way in which the removal of a candidate can lead to the seat being assigned to some other list is if, at some point, more seats are assigned to the list than it has candidates. However, in that case, the candidate would have been elected without being removed, which contradicts the premise. Therefore, the removal of unelected candidates cannot change the results of the election this way either. \square

Lemma 2. *If the number of candidates is equal to number of intended winners, and at least one candidate has received more than zero votes, then at least one candidate has reached the vote quota, provided that the vote quota is lower or equal to the Hare quota without rounding.*

Proof. If no candidates would have reached the vote quota, it follows that each candidate would have fewer votes than the quota. Then, we have

$$t = \sum_{i=1}^w v_i < \sum_{i=1}^w q \leq \sum_{i=1}^w q_{hwr} = \sum_{i=1}^w \frac{t}{w} = w \cdot \frac{t}{w} = t, \quad (4)$$

which is self-contradicting, as v_i is the number of votes received by the i th list and q is the vote quota. In this case, there exists at least one candidate that does reach the vote quota: \square

Lemma 3. *If the number of candidates is greater than the number of intended winners, and at least one candidate has received more than zero votes, then at least one candidate has fewer votes than the vote quota, provided that the vote quota is greater than or equal to the Droop quota without rounding.*

Proof. If no candidates have fewer votes than the vote quota, then it follows that each of c candidates ($c \geq w + 1$) has reached the vote quota, and we have

$$t = \sum_{i=1}^c v_i \geq \sum_{i=1}^c q \geq \sum_{i=1}^c q_{dwr} = \sum_{i=1}^c \left(\frac{t}{w+1} + \varepsilon \right) = \frac{c \cdot t}{w+1} + c\varepsilon. \quad (5)$$

Then, $c \geq w + 1$ can be exploited:

$$t \geq \frac{c \cdot t}{w+1} + c\varepsilon \geq \frac{(w+1) \cdot t}{w+1} + c\varepsilon = t + c\varepsilon, \quad (6)$$

which is self-contradicting, as both c and ε are positive. Therefore, there exists at least one candidate that does not reach the vote quota. \square

In the further analysis, it will be assumed that the vote quota is between the Droop quota without rounding and the Hare quota without rounding, inclusively:

$$q_{hwr} \leq q \leq q_{dwr}. \quad (7)$$

Furthermore, it will be assumed that all methods use the same quota.

Then, it can be demonstrated that the single transferable vote using the Weighted Inclusive Gregory method is equivalent to the largest remainder method when the party lists are sufficiently long:

Theorem 1. *If voters give preferences that coincide with party lists, and the party lists are sufficiently long, then the use of the single transferable vote by the Weighted Inclusive Gregory method results in the election of the same candidates as the use of the largest remainder method. Furthermore, the candidates that are elected using the single transferable vote by the Weighted Inclusive Gregory method before elimination of first candidate are the same as the candidates that are elected under the largest remainder method by quota. In addition, at that moment, the remainder of the votes left for each list, using the largest remainder method, is equal to the number of votes left to the first not-yet-elected candidate using the single transferable vote.*

Proof. The number of candidates from the i th list, elected by quota using the largest remainder method, is

$$e_i = \left\lfloor \frac{v_i}{q} \right\rfloor. \quad (8)$$

As the lists are sufficiently long, the first e_i candidates are elected. Therefore the remainder for each list is

$$r_i = v_i - e_i \cdot q = v_i - \left\lfloor \frac{v_i}{q} \right\rfloor \cdot q. \quad (9)$$

Thus, at first, it is necessary to demonstrate that the same applies to the single transferable vote.

Now, if the quota does not change, using the single transferable vote, each elected candidate is assigned that number of votes. Therefore, the maximal number of candidates elected in such a way is the same, as indicated in Equation (8). Since the lists are sufficiently long, this number can actually be elected. As in each iteration, candidates reaching the vote quota are elected, and this process is repeated until either no candidates reach the vote quota or a sufficient number of candidates is elected. It therefore follows that the number of candidates indicated by Equation (8) will be elected as long as the quota does not change. Furthermore, the number of votes transferred to the first unelected candidate in each list will be equal to the one indicated by Equation (9).

Additionally, the vote quota used in the single transferable vote is not in fact going to change before the elimination of the first candidate, because the vote quota only changes when a vote becomes exhausted (i.e., all candidates involved are either elected or eliminated). However, under the stated conditions, no vote can become exhausted because of the elimination of a candidate, because it is the moment before the first elimination that is being considered here, nor can a vote be exhausted because of the election of all candidates in it, because all party lists are assumed to be sufficiently long. Therefore, the vote quota is not going to change, and the number of elected candidates and number of votes transferred to the first unelected candidate in each list will remain as indicated by Equations (8) and (9).

Therefore, it has been proven that the single transferable vote using the Weighted Inclusive Gregory method elects the same candidates before elimination of the first candidate as the largest remainder method elects by quota. Thus, it is left to show that the single transferable vote elects the same candidates after the elimination of the first candidate as the largest remainder method elects by remainder.

The number of candidates left to be elected is

$$l = w - \sum_{i=1}^N e_i = w - \sum_{i=1}^N \left\lfloor \frac{v_i}{q} \right\rfloor, \quad (10)$$

where N is the number of party lists.

Using the single transferable vote by the Weighted Inclusive Gregory method, if an insufficient number of candidates has been elected without eliminating any candidates, then those candidates with the lowest number of votes will be eliminated. As only the first unelected candidate in each list will have some votes, all candidates below (and the first unelected candidates that received no votes, if any) will be eliminated. As they have no

votes to transfer, no other candidate will receive any additional votes. For the same reason, the vote quota will stay the same. Therefore, after this process ends, each list will have at most one candidate still eligible for election.

At this point, the vote quota is still the same. Thus, none of the candidates can be elected, and the candidate with the least votes will be eliminated. As all candidates further down the list have already been eliminated, no votes can be transferred, and the vote quota will have to be decreased.

The process will be repeated until the intended number of candidates will be elected. As the candidates with the lowest number of votes will be eliminated, l candidates with the most votes can be elected, as with the largest remainder method. It remains to be shown that, under the given conditions, the largest remainder method will not elect more than one candidate from each list and that the single transferable vote will not elect fewer candidates than intended.

When the largest remainder method distributes seats by the remainder, the seats are assigned to party lists one by one in the order of decreasing remainders. If necessary (i.e., if there are still more seats left), the process proceeds cyclically. Therefore, it has to be shown that the number of seats to be distributed by the remainder is going to be lower than or equal to the number of party lists.

The upper limit of l can be found by writing into Equation (10) the Hare quota without rounding:

$$l = w - \sum_{i=1}^N \left\lfloor \frac{v_i}{q} \right\rfloor \leq w - \sum_{i=1}^N \left\lfloor \frac{v_i}{qh} \right\rfloor = w - \sum_{i=1}^N \left\lfloor \frac{v_i}{\frac{t}{w}} \right\rfloor = w - \sum_{i=1}^N \left\lfloor \frac{v_i \cdot w}{\sum_{i=1}^N v_i} \right\rfloor. \quad (11)$$

Then, the further upper bound can be found by removing the rounding:

$$l \leq w - \sum_{i=1}^N \left\lfloor \frac{v_i \cdot w}{\sum_{i=1}^N v_i} \right\rfloor \leq w - \sum_{i=1}^N \left(\frac{v_i \cdot w}{\sum_{i=1}^N v_i} - 1 \right) = w - w + N = N. \quad (12)$$

Thus, the upper bound of the number of seats to be distributed by the remainder is going to be lower than or equal to the number of party lists, and therefore, the largest remainder method will give at most one additional seat by the remainder.

Thus it remains to be shown that the single transferable vote using the Weighted Inclusive Gregory method will result in the election of the same number of candidates. At every point of running the algorithm, either there are more eligible candidates than there are seats left or there are not. If there are more eligible candidates, then by Lemma 3, at least one of them can be eliminated. If there are as many eligible candidates as there are seats, then by Lemma 2, at least one can be elected. Since at each iteration a candidate will be elected or eliminated, and the process will be interrupted after the election of the intended number of candidates, it follows that the right number of candidates will be elected and, therefore, that the single transferable vote using the Weighted Inclusive Gregory method is equivalent to the largest remainder method in this way. \square

However, as the following theorem indicates, Meek's method gives a different result:

Theorem 2. *If the voters give preferences that coincide with the party lists, the party lists are sufficiently long, and the single transferable vote by Meek's algorithm is used, then the candidates from the lists which had more candidates elected already receive more additional votes from vote quota changes. Furthermore, this increase is proportional to the number of candidates from the specific list that have already been elected.*

Proof. Let us consider what happens with a candidate that is after n already elected candidates in the list (receiving v votes) when the vote quota changes from q_1 to q_2 (as the quota can only decrease such that $q_1 > q_2$).

Previously, each of n elected candidates retained q_1 votes, and the first unelected candidate received $v - n \cdot q_1$ votes. After the change in quota, each elected candidate retains q_2 votes, and the first unelected candidate receives $v - n \cdot q_2$ votes. Thus, the number of additional votes for the first still-unelected candidate is

$$(v - n \cdot q_2) - (v - n \cdot q_1) = n \cdot (q_1 - q_2). \quad (13)$$

It can be seen that, as $q_1 - q_2$ is the same for all lists, the number of additional votes is proportional to the number of already elected candidates from the specific list. \square

Thus, we can see that if the vote quota changes (if some candidates that received some votes are eliminated, and their votes are not transferred to other candidates), Meek's method is biased toward the more popular lists compared with the largest remainder method. In fact, that only applies to Meek's method, as implemented by Hill et al. [7,8], since the original Meek's method avoided this predicament by adding all candidates not included in the voter's preference list to a tie for the last place (Such ties were to be processed by splitting the vote into parts representing all possible permutations).

Furthermore, even the single transferable vote using the Weighted Inclusive Gregory method is no longer certain to be equivalent to the largest remainder method when the party lists are truncated.

This can be demonstrated by an example:

Example 1. Let us say that there are three lists: A, consisting of one candidate (A1), B, consisting of 10 candidates (B1, B2, . . . , B10) and C, also consisting of 10 candidates. Let us say that in the election, 10 seats were available, and the lists received, correspondingly, 3000, 300 and 30 votes. Since there are 3330 votes in total, then the Hare quota is 333 votes, and the Droop quota is about 302.7 votes without rounding and 303 with rounding.

Then, using the largest remainder method with any of those quotas, nine seats would be assigned to list A by the quota, and no seats would be assigned by the quota to the other lists. Thus, if list A had enough candidates, only one seat would be distributed by the remainder. However, since list A had only one candidate, that candidate would be elected, while the nine remaining seats would have to be distributed by the remainder. The remainders would be 300 for list B and 30 for list C, and thus distribution would start from list B. Finally, list B would receive five seats, and list C would receive four seats.

Using the single transferable vote by the Weighted Inclusive Gregory method with the rounded Droop quota, the only candidate from list A would be elected, other votes for the list would be lost as non-transferable, and the quota would have to be recalculated. The new quota would be 34 votes. Then, 8 candidates from list B would be elected, leaving 28 votes. Finally, all candidates apart from C1 (having 30 votes) would be eliminated, and C1 would be elected. Thus, list A would receive one seat, list B would receive eight seats, and list C would receive one seat.

Thus, in this case, the difference between the largest remainder method and the single transferable vote is that the single transferable vote also tries to distribute the remaining seats proportionally.

Additionally, it can be demonstrated that the minimally sufficiently long lists do not make the equivalence certain:

Example 2. Let us say that there are three lists that are 15 candidates long (A, B and C, receiving 10,520, 3507 and 1530 votes, respectively) and 3 lists that have just 1 candidate (D, E and F, receiving 1501, 1500 and 1499 votes, respectively). Twenty candidates have to be elected. There are 20057 votes in total, so the Hare quota with rounding is 1002 votes.

Then, using the largest remainder method with the Hare quota, at first (by quota), 10 seats are assigned to list A, 3 seats are assigned to list B, and 1 seat is assigned to each of lists C, D, E and F. That leaves remainders of 500, 501, 528, 499, 498 and 497 votes. Seventeen seats have been distributed by quota, and thus 3 seats are to be distributed by the remainder. They go to lists C, B

and A. Thus, the final distribution is 11 seats for list A, 4 seats to list B, 2 seats to list C and 1 seat each for lists D, E and F. It can be seen that all lists are minimally sufficiently long, for adding more candidates would not give them more seats.

Using the single transferable vote by the Weighted Inclusive Gregory method with the same quota, at first, the first candidates in each list are elected. The remaining votes given for lists D, E and F become non-transferable. Thus, the vote quota has to be changed. The new total number of the remaining votes is 14,045 (9518 in list A, 2505 in list B and 528 in list C), and 14 seats have to be distributed. Thus, the new Hare quota is 896.

Next, 10 additional candidates from list A and 2 candidates from list B are elected. Two seats still have to be assigned, while 558 votes remain for list A, 713 votes remain for list B, and 528 votes remain for list C. As no candidate has enough votes, eliminations start. Eventually, the seats are given to lists B and A.

Thus, using the single transferable vote by the Weighted Inclusive Gregory method, list A receives 12 seats, list B receives 4 seats, and lists C, D, E and F receive 1 seat each. Therefore, list A receives one seat more than it would using largest remainder method, and list C receives one seat less.

It can be seen that in this example, the discrepancy was caused by the population paradox, which is known to affect the largest remainder method.

Naturally, if the preference lists coincide with the party lists, casual vacancies (for example, when one of the elected candidates resigns) can be dealt with by reapplying the same method after removing the candidate in question, just as in the case of the largest remainder method.

2.2. The Case of Preference Lists Starting with Party Lists

Following this, let us consider the case where the preference lists include some other candidates after the list of a given party.

One of the ways in which this can be achieved in practice could include one period when parties would prepare the lists of candidates they propose and another period when parties or groups of voters (not necessarily just the ones proposing any candidates on their own) would prepare the final lists, including candidates proposed by other parties. It seems probable that in most cases, the candidates from the same party would be given in the order indicated by the party, if only because choosing a different order would tend to require more work with relatively little benefit (although it is known that there are some exceptions, especially in the case of single-issue parties [31]). That condition is not required for the further theorems, but by having such similarities, symmetries might simplify things a little and thus will be used in the numerical experiment:

Theorem 3. *If the beginning of the preference list of a group of voters coincides with a party list, then by using the single transferable vote, this list will receive at least as many seats, as it is assigned by the largest remainder method by the quota.*

Proof. Theorem 3 proves this proposition for the case when all party lists are sufficiently long and no candidates are listed after them. However, dropping those conditions cannot result in the election of fewer candidates than the largest remainder method by quota. Shortening other lists can only decrease the vote quota, resulting in the election of more candidates from the list under consideration. Shortening the same list can result in losing seats under the STV, but LRM will then also give fewer seats, so the number will stay the same. Finally, adding other candidates after the lists cannot affect the series of candidates that already reach the vote quota. □

Theorem 4. *If the beginning of the preference list of a group of voters coincides with a party list that is minimally sufficiently long, and the Weighted Inclusive Gregory method is used, then the number of votes transferred from this group of voters to other candidates is going to be less than the original vote quota.*

Proof. Let us assume the contrary; that is, there can be a case when the number of votes transferred in such a way is equal to the original vote quota or higher. Then, if another candidate was inserted into the end of the party list, that candidate would receive enough votes to reach the vote quota (which at that point would be lower or equal to the original quota) and would be elected. However, the party list was said to be minimally sufficiently long, which implies that adding another candidate could not lead to more seats being assigned to the list. That is a contradiction, and thus the assumption is wrong. \square

This indicates that the number of seats changing hands because of adding further preferences is going to be limited.

In fact, as the further theorem indicates, the influence of such transferred votes is going to be even more limited:

Theorem 5. *If each preference list begins with a party list followed by zero or more candidates from other party lists in some order, and all party lists are sufficiently long, then for each list, the single transferable vote by the Weighted Inclusive Gregory method or Meek's algorithm assigns either the same number of seats, as assigned by the largest remainder method by quota, or one seat more.*

Proof. If the further priorities (after the first party list) were to be removed, then by Theorem 1, the single transferable vote using the Weighted Inclusive Gregory method would assign to each list the same number of seats before elimination of the first candidate, as with the largest remainder method. The vote quota would not change, and no votes would be transferred to candidates that already had some votes assigned to them for being elected (the party lists have no candidates in common), and thus the same would be true for Meek's algorithm.

After this step, all remaining votes would have been transferred to the first unelected candidate from each list. Such a candidate necessarily exists, for the lists are said to have at least one more candidate than is necessary for them to be minimally sufficiently long. As no votes can be transferred further at that moment, the same will be true if there are more candidates in the priority lists after the first party list.

At that point, no unelected candidate has reached the vote quota, and therefore, if any seats are left unfilled, some candidates will have to be eliminated. At first, all candidates without transferred votes would be eliminated (As they have no transferred votes, they cannot transfer votes further). Thus, only the first unelected candidates from each party list would be left. As each of them can eventually either be elected or unelected, each list would be assigned either the seats it already has at that point (which would be equal to the number of seats assigned by the largest remainder method by quota) or one seat more. \square

By itself, this result might be undesirable, as the transfers of votes are not used fully. (It could be expected that elimination of several candidates which are followed by the same party list might result in two or more of its candidates being elected). However, of course, if the goal is to achieve results similar to the results of the largest remainder method, then this result will be desirable.

In order to let the transfers of votes be used more fully, the way in which candidates are eliminated can be changed. For example, it might be sufficient to reintroduce the candidates eliminated previously after a candidate is elected (RI), as proposed, for example, in [10] (other suggestions given there—the use of the Hare quota and trying out the elimination of various sets of candidates—seem to be unnecessary in this specific case). That seems to be the most straightforward (although not necessarily the most efficient) way. Other promising methods would include handling the elimination of candidates in a more complex way, as performed, for example, in the methods described in [3,9,14,15].

Then, it is possible to find the lower and upper bounds for the number of seats assigned with transfers. First, let us prove a lemma:

Lemma 4. *If each preference list begins with a party list followed by zero or more candidates from other party lists in some order, and all party lists are sufficiently long, then by using the single transferable vote by the Weighted Inclusive Gregory method, the same list cannot receive two seats in a row after elimination of the first candidate. Furthermore, the same applies both with reintroduction of the eliminated candidates and without it.*

Proof. Let us consider a point right before the list is going to obtain enough votes to earn a seat. At that point, there are n other lists that have not been eliminated (Theorem 5 indicates that only one candidate per list can be left at that point), the vote quota is q , and the list receiving the next seat has v votes left. Let us order the rest of the lists in ascending order by the number of votes. As the stage when the elimination of candidates is possible has been reached, at this point, all lists have fewer votes than the vote quota:

$$v_1 \leq v_2 \leq \dots \leq v_i \leq \dots \leq v_n < q. \quad (14)$$

The same has to be true for the list winning the next seat:

$$v < q. \quad (15)$$

Since this list is going to win the next seat but has not won it already, the candidates from the list with the lowest number of votes (v_1) are eliminated, and their votes are added to the votes of this list, which is enough to reach the vote quota:

$$v + v_1 \geq q. \quad (16)$$

After winning this seat, the numbers of votes for those lists (including the list that won that seat) are reduced proportionally, and thus their order does not change. Therefore, at the next step, the lists for which the votes were transferred to the list that won the seat in the previous step will be eliminated again (or stay eliminated, if eliminated candidates are not reintroduced), and in the same order, the votes will be transferred to the same list (If they had a candidate from a different list right after the last elected candidate, that only makes it harder for the list to get a second seat in a row). Let us assume that no other lists that were eliminated at a previous step are going to avoid elimination in this step (otherwise, the list that won the previous seat can only be eliminated earlier, thus still not receiving two seats in a row). The numbers of votes are reduced in such way that the total number of votes received by the list that won the previous seat is reduced by the vote quota. Then, using Equations (15) and (14), we can estimate the number of votes left for the list that won the previous seat:

$$v + v_1 - q < q + v_1 - q = v_1 \leq v_2. \quad (17)$$

As we can see, the candidate from the list that won the previous seat does not reach the vote quota and has the least number of votes from the candidates that are still not eliminated. Therefore, the list that won the seat is going to be eliminated next and will not receive the second seat in a row. \square

Then, we can estimate the upper and lower bounds themselves. We will assume that, when applying the largest remainder method, the votes given to the preference list go to the first party list on it. The bounds apply for both the case with the reintroduction of eliminated candidates and the case without such a reintroduction.

Theorem 6. *If each preference list begins with a party list followed by zero or more candidates from other party lists in some order, and all party lists are sufficiently long, then for each list, the single transferable vote by the Weighted Inclusive Gregory method cannot assign fewer seats than the largest remainder method would assign by quota to that list. Furthermore, the candidates which would have been elected by the quota in such way are also elected using the Weighted Inclusive Gregory method. Additionally, the Weighted Inclusive Gregory method cannot assign more seats*

than the sum of the number of seats that the largest remainder method would assign by quota to the same list and the rounded up half of the total number of seats that the the largest remainder method would distribute by the remainder to all party lists.

Proof. At first, the candidates that reach the quota are elected. Since at that stage, no candidates can be eliminated, and the party lists are said to be sufficiently long, the preferences listed after the first party list cannot affect the distribution of seats at this stage. Therefore, by Theorem 1, at this stage, each list would receive the number of seats that the largest remainder method would distribute by quota. That establishes the lower bound and demonstrates that the same candidates would be elected. As no candidates have been eliminated at this stage, the reintroduction of eliminated candidates does not change anything here.

As the largest remainder method distributes the seats by quota and by remainder, only seats distributed by the remainder are left after this stage. Thus, the upper bound can be found when all those seats are assigned to the same list.

However, this does not mean that all seats assigned by the remainder can go to the same list. By Lemma 4, no list can receive two of such seats in a row. Thus, no more than half of them (rounded up) have to be added. \square

It should be noted that in the case of the Weighted Inclusive Gregory method without the reintroduction of eliminated candidates, the upper bound mentioned in Theorem 6 is going to be weaker than the upper bound mentioned in Theorem 5 unless all seats (or all seats but one) are distributed by the quota.

If the preference lists start with party lists, casual vacancies can also be dealt with by reapplying the same method after removing the candidate in question, just as in the case of preference lists coinciding with party lists.

2.3. The Case with an Electoral Threshold

Let us consider the ways to implement the electoral threshold for the single transferable vote. The most obvious way is to remove the votes cast for the lists under the threshold (which is usually given as a percentage of votes but might also be a set number of votes):

Theorem 7. *If the voters give preferences that coincide with the party lists, and those party lists are sufficiently long, then the use of the single transferable vote using the Weighted Inclusive Gregory method after removal of the votes given for the party lists below the electoral threshold results in the election of the same candidates as with the use of the largest remainder method with the same electoral threshold.*

Proof. As the same votes are removed for both methods, the numbers of remaining votes and the remaining party lists are the same in the cases for both methods. Therefore, by Theorem 1, the results of those methods are going to be equivalent. \square

However, while removing votes is the most obvious way to implement the electoral threshold, it might not be the most suitable way, as it results in a significant amount of votes being wasted. This is inevitable when using methods based on party lists (and it is with those methods that electoral thresholds are currently used), because in such cases, the voters who voted for lists below the electoral threshold have provided no other information. However, that is not necessarily the case when using the single transferable vote. Thus, it is possible to implement the electoral threshold in a different way: by removing not the votes but the candidates. Then, the simple way to implement the electoral threshold would be by removing the candidates that were not in the first w places (where w is the number of candidates to be elected) with at least the number of votes corresponding to the electoral threshold. Let us show that this method achieves the same result when only party lists are used:

Theorem 8. *If the voters give preferences that coincide with the party lists, and those party lists are sufficiently long, then if those candidates are removed, who are within the first w places (where w is the number of candidates to be elected) and within fewer votes than the number of votes corresponding to the electoral threshold, then the use of the single transferable vote using the Weighted Inclusive Gregory method results in the election of the same candidates as with the use of the largest remainder method with the same electoral threshold.*

Proof. As the party lists have no candidates in common, the candidates in the first w places in each list will be removed if and only if the list is below the electoral threshold. In addition, all candidates below the first w places in their list will be removed.

None of those candidates could have been elected using the largest remainder method, for no candidate in a list below the electoral threshold can be elected, and no more than w candidates from each list can be elected. Therefore, by Lemma 1, removal of those candidates could not change the result when using the largest remainder method, and thus such removal is equivalent to the application of electoral threshold. Thus, by Theorem 1, the Weighted Inclusive Gregory method will also elect those same candidates. \square

It should be noted that such use of the electoral threshold complicates the application of Theorems 5 and 6, for it would have to be applied after elimination of the candidates below the threshold. Therefore, votes given to lists failing to reach the electoral threshold can still help a further list to gain two or more additional seats.

2.4. Algorithm for Using the STV with Preference Lists Based on Party Lists

Theorems 6–8 suggest a simple procedure of finding the results of the single transferable vote if the preference lists are based on party lists:

1. Preprocess the lists (remove candidates elected in other ways, apply the electoral threshold, etc.);
2. Group the votes by the party lists at the beginning of the preference lists;
3. Apply the largest remainder method to the party lists at the beginning of the preference lists;
4. Distribute the seats that the largest remainder method assigned by quota up to the end of the party lists;
5. Update the vote counts by subtracting the vote quota for each elected candidate (distributing the subtracted votes proportionally among the grouped votes) and the candidate lists by removing the elected candidates;
6. If any votes ended up with no eligible candidates left, then update the vote quota;
7. If there was a party list that had all its candidates elected, return to step 2;
8. If reintroduction of the candidates is not used, then remove all candidates with the exception of the first unelected candidate in each party list. If reintroduction of the candidates is used, then also leave one candidate per list for each of the two seats that remain to be filled at this point;
9. Distribute the remaining seats (if any) by the single transferable vote.

Step 7 makes it easier to deal with preferences for having an independent candidate followed by party lists. If an independent candidate is elected, then it is possible to use the largest remainder method afterward. The resulting loop ensures that steps 8 and 9 would be applied to party lists that are sufficiently long.

Theorems 4 and 6 indicate that only a limited number of votes and seats would be left to distribute after step 4. That, in turn, limits the time required for step 9.

Theorem 2 indicates that steps 5 and 6 would have to be more complicated in the case of Meek's method.

It is also possible to add a condition to step 4 so that it would be skipped if some lists are not sufficiently long. That might make a difference if the vote quota can change while there are candidates that have reached the vote quota (as in Example 2), and there are some specific rules concerning this case.

Ordinarily, implementing the STV by the Weighted Inclusive Gregory method for an election with c candidates and v voting patterns would require $O(c)$ removals of individual candidates (either elected or eliminated). Each of such removals would require going through all voting patterns (which can include every candidate at first). Thus, the time complexity of the ordinary application of the STV would be $O(c^2v)$.

Ordinarily, the STV implemented by the Weighted Inclusive Gregory method with the reintroduction of eliminated candidates for an election with c candidates, v voting patterns and w candidates to be elected would require $O(w)$ removals of elected candidates. Each such removal would require $O(c - w)$ eliminations and reintroductions, which would require going through all voting patterns (at first including all candidates) to recalculate the votes. Thus, the time complexity of such an application of the STV would be $O(w(c - w)cv)$.

Using the proposed procedure, the STV by the Weighted Inclusive Gregory method for an election with p sufficiently long party lists having l candidates each, with r possible voting patterns per party list, would require $O(pr)$ divisions for step 3. This leads to $O(pl)$ removals of elected and eliminated candidates. Finally, step 9 would deal with p candidates and would have a time complexity $O(p^3r)$. Thus, the time complexity of the proposed procedure with the Weighted Inclusive Gregory method would be $O(prl + p^3r)$, and thus this leads to an improvement over ordinary implementation, which would have a time complexity of $O(c^2v) = O(p^3rl^2)$.

Using the proposed procedure, the STV by the Weighted Inclusive Gregory method with the reintroduction of eliminated candidates for an election with p sufficiently long party lists having l candidates each, with r possible voting patterns per party list, would also require $O(pr)$ divisions for step 3. This leads to $O(prl)$ removals of elected and eliminated candidates. Finally, step 9 would deal with $O(p^2)$ candidates (assuming that the party lists are long enough) and $O(p)$ seats. Therefore, it would have a time complexity $O(p^6r)$. Thus, the time complexity of the proposed procedure with the Weighted Inclusive Gregory method would be $O(prl + p^6r)$. Therefore, this leads to an improvement over ordinary implementation, which would have time complexity of $O(w(c - w)cv) = O(w(pl - w)p^2rl) = O(wp^3rl^2 - w^2p^2rl)$, provided that the number of party lists is relatively small.

It should be noted that, for the sake of this algorithm, "party list" is understood to be a group of candidates that is listed in the same order without gaps in all preference lists, especially when it is at the beginning of the list (and it is such commonalities, a sort of symmetries, that are being exploited here). Therefore, if a single party with two internal factions would offer two preference lists, each listing all the candidates of one faction before the candidates of another, those two lists of candidates would count as party lists for the sake of this algorithm. If the same party would offer a third preference list where the candidates of those factions would be listed in alternating order, the algorithm, strictly speaking, would be inapplicable (or all those candidates would count as separate party lists, which would be computationally inefficient).

Of course, there are situations where the algorithm could still be applied (or be adapted to apply) even where candidates from party lists are not listed in the same order in all available preference lists. For example, it could happen that the preference list causing the problems would receive no votes. It could be that differences in the order of candidates would happen in the part of the preference lists that is only processed in step 9 (although it would be necessary to deal with all those orders in step 8). Additionally, it is possible to apply the algorithm in cases where there are two preference lists, of which the second is an inversion of the first, and treat them as party lists (although they would have all candidates in common). It could be possible to apply the algorithm if the first party list is given in several ways by treating each variant as a separate party list and electing candidates that would be elected in at least one of them (but if the same candidate would be elected in more than one list, that would require a more complex distribution of votes in step 5). Still, it is unclear if such adaptations would be worth the computational cost.

Also, the time complexities indicate that it is the reduction of the number of party lists that is especially important for keeping the calculation time low.

3. Numerical Experiments

The theoretical investigation can demonstrate the equivalence of those voting methods under certain conditions. However, it is also important to know what can be expected to happen in more realistic conditions, especially when equivalence fails to hold. In such cases, it is important to find out how significant the difference would be.

For this, three numerical experiments were performed. In the first, the case when the preference lists coincide with party lists in an actual election was considered. This made it possible to check how significant the difference was between Meek's algorithm and the Weighted Inclusive Gregory method (mentioned in Theorem 2) under realistic conditions. In the second, a case with a pre-election coalition was considered. This made it possible to observe how much things would change if the parties used the features of the single transferable vote in the most straightforward way. In the third, preferences consisting of permutations of party lists were considered. This made it possible to check how significant the changes would be if the parties would use the features of the single transferable vote more extensively. These three numerical experiments corresponded to three strategies: when parties did not include other candidates in their preference lists, when parties included other candidates from intended coalition partners in their preference lists and when parties included all other candidates in their preference lists. Out of these three scenarios, the third seems to be the least realistic, given that in the actual elections using the STV, political parties tend to avoid telling their voters to let their votes to be transferred to candidates from other parties [32].

As it is the equivalence with the largest remainder method that is being investigated, these numerical experiments require some data from an election performed using proportional representation with party lists (Constructing such lists from, let us say, preference lists given by voters in an election using the single transferable vote would be harder). Furthermore, it would be convenient if the election would involve relatively many seats and relatively few lists. Thus, the data from the 2020 elections of Seimas (Lithuanian Parliament) in a multi-member district were used (Table 1). There was an additional complication of some candidates being elected in single-member districts, but that was ignored for this experiment, as all lists were sufficiently long even after removal of those candidates. Preference votes (up to five by each voter) would also be ignored to simplify the experiment. Naturally, it could be expected that the behavior of the voters would change if other voting methods were used. However, it could still be expected that the voters would still consider the case when their behavior did not change.

The electoral threshold was 5%. The vote share of list number 8 was actually 4.974%, which was right below that threshold.

It should be noted that ranking all 1724 candidates (perhaps even 70 candidates, with one for each seat) individually would be slow and inconvenient for voters, and counting such votes would also be inconvenient for electoral officials. Thus, it seems that the use of preference lists prepared by parties, candidates or groups of voters before the election is a much more practical way to implement the single transferable vote in such cases.

First, let us look at how the use of a variety of methods might have changed the results of this election (Table 2). It should be noted that in the case of the single transferable vote, the electoral threshold was applied by the removal of candidates (as mentioned in Theorem 8). In addition, the Droop quota without rounding was used (as it is more common for the single transferable vote).

Table 1. The results of the 2020 elections of Seimas in a multi-member district [33].

List	Candidates	Votes	By Quota	Remainder	By Rem.	Total
1	47	13,337 (1.2%)	0	-	0	0 (0%)
2	121	23,355 (2.0%)	0	-	0	0 (0%)
3	74	107,093 (9.4%)	8	3869	0	8 (11.4%)
4	33	2946 (0.3%)	0	-	0	0 (0%)
5	141	292,124 (25.8%)	22	8258	1	23 (32.9%)
6	94	26,769 (2.4%)	0	-	0	0 (0%)
7	69	25,098 (2.2%)	0	-	0	0 (0%)
8	140	56,386 (5.0%)	0	-	0	0 (0%)
9	132	37,197 (3.3%)	0	-	0	0 (0%)
10	41	5808 (0.5%)	0	-	0	0 (0%)
11	48	11,352 (1.0%)	0	-	0	0 (0%)
12	141	79,755 (7.0%)	6	2337	0	6 (8.6%)
13	141	204,791 (18.1%)	15	11,246	1	16 (22.9%)
14	125	19,303 (1.7%)	0	-	0	0 (0%)
15	98	8825 (0.8%)	0	-	0	0 (0%)
16	140	110,773 (9.8%)	8	7549	1	9 (12.9%)
17	139	108,649 (9.6%)	8	5425	0	8 (11.4%)
Total	1724	1,133,561 (100%)	67	38,684	3	70 (100%)

Table 2. Distribution of seats in the 2020 elections of Seimas in a multi-member district, if held using a variety of methods.

List	LRM + Th	LRM	STV-WIG	STV-M	STV-WIG + Th	STV-M + Th
1	0	1	1	1	0	0
2	0	1	1	1	0	0
3	8	7	7	7	8	8
4	0	0	0	0	0	0
5	23	18	18	19	23	23
6	0	2	2	2	0	0
7	0	2	2	1	0	0
8	0	3	3	4	0	0
9	0	2	2	2	0	0
10	0	0	0	0	0	0
11	0	1	1	0	0	0
12	6	5	5	5	6	6
13	16	13	13	13	16	16
14	0	1	1	1	0	0
15	0	0	0	0	0	0
16	9	7	7	7	9	9
17	8	7	7	7	8	8
Total	70	70	70	70	70	70

It can be seen that, just as Theorem 2 stated, by using Meek's method, the more popular lists gained seats from the less popular lists; lists 5 and 8 gained a seat, while lists 7 and 11 lost a seat. However, the difference ended up being relatively insignificant and vanished when using the electoral threshold.

Still, using the single transferable vote, parties would get an option to provide priority lists including candidates from other parties, and it is probable that at least in some cases (for example, pre-election coalitions), such an option would be used. To explore what would be changed by such a possibility, let us consider a case (corresponding to an actual pre-election coalition created for this election) where list 3 indicated lists 12 and 5 next, list 5 indicated lists 12 and 3 next and list 12 indicated lists 3 and 5 next. Table 3 gives the distribution of seats in such a case.

It can be seen that with the pre-election coalition with the electoral threshold, list 3 (one of the lists in a coalition) gained a seat, and list 16 (one of the lists not in a coalition) lost a seat. Originally (Table 1), list 16 gained that extra seat by the remainder. Using the single transferable vote made it possible for the coalition to have that extra seat transferred to a list in the coalition. Yet, it might seem strange that this seat did not go to another list from the coalition—list 12—which, after all, was listed as second by both other lists. This was because by the time the votes could be transferred from list 5, the candidate on list 12 had already been eliminated. It can be seen that the reintroduction of the eliminated candidates after the election of a candidate corrected this anomaly, and the seat went to list 12. One of the lists from the coalition—list 5—also gained a seat by the remainder. This, along with Theorem 5, explains why this list did not gain another seat here.

However, we can see that the coalition changed nothing if the electoral threshold was not used. Thus, while pre-election coalitions can benefit from such cooperation, the difference is not likely to be drastic unless one of those lists fails to reach the threshold.

Table 3. Distribution of seats in the 2020 elections of Seimas in a multi-member district if held under a variety of conditions and there were a pre-election coalition represented in the priority lists.

List	STV-WIG	STV-M	STV-WIG + Th	STV-M + Th	STV-WIG + Th + RI
1	1	1	0	0	0
2	1	1	0	0	0
3	7	7	9	9	8
4	0	0	0	0	0
5	18	19	23	23	23
6	2	2	0	0	0
7	2	1	0	0	0
8	3	4	0	0	0
9	2	2	0	0	0
10	0	0	0	0	0
11	1	0	0	0	0
12	5	5	6	6	7
13	13	13	16	16	16
14	1	1	0	0	0
15	0	0	0	0	0
16	7	7	8	8	8
17	7	7	8	8	8
Total	70	70	70	70	70

The next numerical experiment was performed to see to what extent the results could be changed by the transferring of votes. As the number of possible permutations of lists was far too high, a sample of 1000 groups of random permutations was tried out (see Table 4). Additionally, estimates of the minimal and maximal results were made by investigating the cases where a specific list was given as second or last in the permutations after all other lists. (The other lists were sorted cyclically. Even if that was not certain to guarantee the true minimum and true maximum, it was likely to stay close).

Table 4. Distribution of seats in the 2020 elections of Seimas in a multi-member district if held under a variety of conditions and with priority lists, including permutations of lists of other parties. Results are shown in the form of min-median-max.

List	STV-WIG	STV-WIG + RI	STV-M	STV-WIG + Th	STV-M + Th
1	0-1-1	0-1-2	0-1-1	0-0-1	0-0-1
2	1-1-2	1-1-3	1-1-2	0-0-15	0-0-15
3	6-7-7	6-7-7	6-7-7	6-8-21	6-8-21
4	0-0-0	0-0-2	0-0-0	0-0-1	0-0-1
5	18-18-19	18-18-19	18-18-19	18-20-33	18-20-33
6	1-2-2	1-2-2	1-2-2	0-0-15	0-0-15
7	1-2-2	1-2-2	1-2-2	0-0-14	0-0-15
8	3-3-4	3-3-5	3-3-4	0-0-15	0-0-15
9	2-2-3	2-2-3	2-2-3	0-0-15	0-0-15
10	0-0-1	0-0-1	0-0-1	0-0-1	0-0-1
11	0-1-1	0-1-1	0-1-1	0-0-1	0-0-1
12	4-5-5	4-5-6	4-5-5	4-7-20	4-7-20
13	12-13-13	12-13-14	12-13-13	12-14-27	12-14-27
14	1-1-2	1-1-2	1-1-2	0-0-1	0-0-1
15	0-0-1	0-0-1	0-0-1	0-0-1	0-0-1
16	6-7-7	6-7-8	6-7-7	6-8-21	6-9-21
17	6-7-7	6-7-8	6-7-7	6-8-21	6-8-21

It should be noted that with the electoral threshold, the Weighted Inclusive Gregory method found the same results both with the reintroduction of the eliminated candidates and without it.

It can be seen that, as Theorem 5 indicated, when using the Weighted Inclusive Gregory method without the electoral threshold, the minimal and maximal values did not differ by more than one. The same applied to the case with Meek's method.

Using the Weighted Inclusive Gregory method with the reintroduction of the eliminated candidates made greater differences between the maximal and minimal values possible, and for 8 lists out of 17, this difference was 2. Thus, as Theorem 4 indicated, without the electoral threshold, the transfers were not going to affect a great number of seats. When using the largest remainder method without the electoral threshold for the party lists (Table 2), nine seats were distributed by the remainder. Thus, Theorem 6 indicates that the upper bound of the difference for each list was five.

With the electoral threshold, the differences between the minimal and maximal number of seats were much greater, as slightly more than 5% of the votes were given to lists that had less than 70 candidates, and thus if a list were first or second in all the votes, at least one candidate from it was going to pass the threshold. In several cases, the maximum was one because one of the lists had 69 candidates, and thus in those cases, just one candidate passed the threshold. A further reason for such an increase in these difference is that there were more votes that could be transferred because all the votes from the lists that did not reach the threshold were transferred.

The medians were rather similar to the results without transfers. They were somewhat lower than the results without transfers for the more popular lists. This was something that could be expected, for even if no additional candidates passed the electoral threshold, the more popular lists needed a higher proportion of transferred votes to keep the same number of seats, and they tended to not receive as many votes, as the lists were positioned randomly.

Table 5 shows the calculation times for the previously mentioned scenarios. In all cases the calculations were performed 20 times using GNU Octave v. 6.20 by The Octave Project Developers, and i5-4570 CPU.

Table 5. Calculation times for various scenarios in seconds (means and standard deviations).

Scenario	STV-WIG	STV-WIG + RI	STV-WIG New	STV-WIG + RI New
Lists	1.3301 ± 0.0164	8.0403 ± 0.1016	0.7036 ± 0.0035	4.3562 ± 0.0062
Lists Th	1.4342 ± 0.0187	2.7834 ± 0.0109	0.6634 ± 0.0019	1.3103 ± 0.0024
Coalition	1.4160 ± 0.0077	8.3942 ± 0.0268	0.7006 ± 0.0040	4.3781 ± 0.0171
Coalition Th	1.4302 ± 0.0237	2.7313 ± 0.0491	0.6610 ± 0.0016	1.3088 ± 0.0040
Random	1.5373 ± 0.0520	11.3648 ± 1.4569	0.6952 ± 0.0033	5.8996 ± 0.7450
Random Th	2.8216 ± 0.2112	5.6733 ± 1.4137	0.5737 ± 0.2814	1.2283 ± 0.7376

It can be seen that in all those cases, use of the proposed method did make the calculations faster, usually about two times faster. It can be seen that, as expected, the calculations with the reintroduction of eliminated candidates required more time. It can be seen that with reintroduction of the eliminated candidates, the calculations took much less time when the electoral threshold was applied. It was rather unexpected that the calculations using the ordinary implementation of the STV took more time when the electoral threshold was used. One reason for this is that application of the electoral threshold takes more time when preference lists are longer. (Applying the electoral threshold is much faster when it is possible to take existence of separate party lists into account).

4. Discussion

It has been demonstrated that, under certain conditions, the single transferable vote can be adapted to achieve the same results as those with the largest remainder method while still making it possible to use the features of the single transferable vote, for example, straightforward handling of independent candidates.

Some of the necessary conditions, such as a sufficient number of candidates in each list, are likely to hold. The condition that is least likely to hold would seem to be that of the behavior of voters and parties staying the same.

In fact, the changes in behavior, which can be caused by having an option to list candidates of other parties, seem to be hard to predict. For example, on the one hand, voters who would avoid voting for a party that risks ending up below the electoral threshold might choose to vote for it if the votes could be transferred to another party they like. The possibility that this might move the party above the electoral threshold could encourage the party to list candidates of other parties in the suggested preference list. On the other hand, some voters might avoid voting for a party they would otherwise vote for if, in such a case, their votes could be transferred to a party they disliked.

The exploration of such possible changes in behavior seems to be one possibility for future research directions. Another research direction would consist of finding ways to adapt the proposed algorithm for using the STV with preference lists based on party lists to more cases.

By themselves, the findings of this paper are most easily applied to a voting method where voters choose among preference lists prepared beforehand and based on party lists, as illustrated in the numerical experiment (somewhat similar to what was proposed in [20], where it was proposed to have arbitrary permutations possible, which seems to make vote counting somewhat too complex, and to have a different way of calculating the results, which would seem to be impractical in cases with larger numbers of seats and candidates). In such cases, it is most likely that those lists would be prepared by political parties (or groups of voters) in two stages. In the first stage, the parties or groups would select the candidates they propose, and in the second stage, they would list the proposed candidates (all or just some) in order of priority. It is also shown that under certain conditions, such a method would achieve the same results as the largest remainder method (and that it is still likely to achieve relatively similar results under some other realistic conditions) while, for example, making it possible to reduce the number of wasted votes, just as with the single

transferable vote. Furthermore, it looks like this would be much faster than the ordinary implementation of the single transferable vote.

However, investigation of the use of the single transferable vote in cases where priorities are based on party lists seems to be rather important, even if other kinds of priorities are possible. For example, it is known that under the single transferable vote, parties tend to ask the voters in various areas to list the candidates in a different order [32,34]. That might seem surprising, for that is a clear example of strategic voting, and the single transferable vote has been shown to be resistant to strategic voting [35]. Therefore, Theorem 5 explains why a more straightforward strategy of asking the voters to use the same party lists is suboptimal. Such analysis can also suggest that the reintroduction of eliminated candidates after the election of a candidate (or some other methods) might avoid this difficulty, thus simplifying the work of local party organizations (maybe making things easier for the voters as well) while, of course, making counting votes more complex. Likewise, the use of the electoral threshold applied to separate candidates (see Theorem 8) can also be applied in elections by the single transferable vote even if party lists are not prepared, should that be found to be advisable.

5. Conclusions

The investigation showed that the single transferable vote by the Weighted Inclusive Gregory method is certain to achieve the same results as the largest remainder method, provided that each party list has enough candidates. Furthermore, it has been shown that it is possible to apply the electoral threshold to the single transferable vote by removing the candidates who fail to reach a sufficiently high position (corresponding to the number of candidates to be elected) in a sufficient number of votes (corresponding to the level of the electoral threshold). This indicates that the Weighted Inclusive Gregory method is a suitable replacement for the largest remainder method when it is desirable to avoid wasted votes and to handle independent candidates in a straightforward way, but it is also desirable to keep the results as close to the ones achieved under the largest remainder method as possible. It seems that, as far as voters and political parties are concerned, the procedure would stay mostly the same, and the lists would only have to be made in two stages. During the first, the candidates would be proposed (perhaps in the way corresponding to proposing the party lists under the largest remainder method), and during the second, preference lists ranking the proposed candidates would be suggested (perhaps by ranking the party lists and keeping the candidate order in each list). An algorithm to simplify application of the single transferable vote for such a case was proposed. It has been shown that such an algorithm makes the calculations faster than the use of ordinary implementations of the single transferable vote.

Furthermore, the investigation has shown that common implementations of the single transferable vote limits the transfer of votes between lists. One way to avoid such a limitation is the reintroduction of eliminated candidates each time a candidate reaches the vote quota and is elected.

The investigation has also shown that the single transferable vote by Meek's method is not certain to achieve the same results as the largest remainder method, for it does not distribute the non-transferable votes equally between the remaining lists, being biased toward more popular lists.

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Abbreviations

The following abbreviations are used in this manuscript:

LRM	Largest remainder method
RI	Reintroduction of eliminated candidates after election of a candidate
STV	Single transferable vote
STV-M	Single transferable vote by Meek's method
STV-WIG	Single transferable vote by Weighted Inclusive Gregory method
Th	Electoral threshold

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